Analytic Geometry in Three Dimensions

11.1 The Three-Dimensional Coordinate System
11.2 Vectors in Space
11.3 The Cross Product of Two Vectors
11.4 Lines and Planes in Space

*In Mathematics*
A three-dimensional coordinate system is formed by passing a $z$-axis perpendicular to both the $x$- and $y$-axes at the origin. When the concept of vectors is extended to three-dimensional space, they are denoted by ordered triples $v = (v_1, v_2, v_3)$.

*In Real Life*
The concepts discussed in this chapter have many applications in physics and engineering. For instance, vectors can be used to find the angle between two adjacent sides of a grain elevator chute. (See Exercise 62, page 839.)

**IN CAREERS**
There are many careers that use topics in analytic geometry in three dimensions. Several are listed below.

- Architect
  Exercise 77, page 816
- Geographer
  Exercise 78, page 816
- Cyclist
  Exercises 61 and 62, page 830
- Consumer Research Analyst
  Exercise 61, page 839
The Three-Dimensional Coordinate System

Recall that the Cartesian plane is determined by two perpendicular number lines called the \( x \)-axis and the \( y \)-axis. These axes, together with their point of intersection (the origin), allow you to develop a two-dimensional coordinate system for identifying points in a plane. To identify a point in space, you must introduce a third dimension to the model. The geometry of this three-dimensional model is called solid analytic geometry.

You can construct a three-dimensional coordinate system by passing a \( z \)-axis perpendicular to both the \( x \)- and \( y \)-axes at the origin. Figure 11.1 shows the positive portion of each coordinate axis. Taken as pairs, the axes determine three coordinate planes: the \( xy \)-plane, the \( xz \)-plane, and the \( yz \)-plane. These three coordinate planes separate the three-dimensional coordinate system into eight octants. The first octant is the one in which all three coordinates are positive. In this three-dimensional system, a point \( P \) in space is determined by an ordered triple \((x, y, z)\), where \( x \), \( y \), and \( z \) are as follows.

\[
\begin{align*}
  x &= \text{directed distance from } yz\text{-plane to } P \\
  y &= \text{directed distance from } xz\text{-plane to } P \\
  z &= \text{directed distance from } xy\text{-plane to } P 
\end{align*}
\]

A three-dimensional coordinate system can have either a left-handed or a right-handed orientation. In this text, you will work exclusively with right-handed systems, as illustrated in Figure 11.2. In a right-handed system, Octants II, III, and IV are found by rotating counterclockwise around the positive \( z \)-axis. Octant V is vertically below Octant I. Octants VI, VII, and VIII are then found by rotating counterclockwise around the negative \( z \)-axis. See Figure 11.3.
Section 11.1 The Three-Dimensional Coordinate System

Example 1 Plotting Points in Space

Plot each point in space.

a. (2, -3, 3)  

b. (-2, 6, 2)  

c. (1, 4, 0)  

d. (2, 2, -3)

Solution

To plot the point notice that and To help visualize the point, locate the point in the $xy$-plane (denoted by a cross in Figure 11.4). The point lies three units above the cross. The other three points are also shown in Figure 11.4.

Now try Exercise 13.

The Distance and Midpoint Formulas

Many of the formulas established for the two-dimensional coordinate system can be extended to three dimensions. For example, to find the distance between two points in space, you can use the Pythagorean Theorem twice, as shown in Figure 11.5. Note that $a = x_2 - x_1$, $b = y_2 - y_1$, and $c = z_2 - z_1$.

Distance Formula in Space

The distance between the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ given by the Distance Formula in Space is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$  

Example 2 Finding the Distance Between Two Points in Space

Find the distance between (1, 0, 2) and (2, 4, -3).

Solution

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$  

Distance Formula in Space

$$= \sqrt{(2 - 1)^2 + (4 - 0)^2 + (-3 - 2)^2}$$  

Substitute.

$$= \sqrt{1 + 16 + 25}$$  

Simplify.

$$= \sqrt{42}$$  

Simplify.

Now try Exercise 27.

Notice the similarity between the Distance Formulas in the plane and in space. The Midpoint Formulas in the plane and in space are also similar.

Midpoint Formula in Space

The midpoint of the line segment joining the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ given by the Midpoint Formula in Space is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$
Example 3  Using the Midpoint Formula in Space

Find the midpoint of the line segment joining \((5, -2, 3)\) and \((0, 4, 4)\).

Solution

Using the Midpoint Formula in Space, the midpoint is

\[
\left( \frac{5 + 0}{2}, \frac{-2 + 4}{2}, \frac{3 + 4}{2} \right) = \left( \frac{5}{2}, 1, \frac{7}{2} \right)
\]

as shown in Figure 11.6.

The Equation of a Sphere

A sphere with center \((h, k, j)\) and radius \(r\) is defined as the set of all points \((x, y, z)\) such that the distance between \((x, y, z)\) and \((h, k, j)\) is \(r\), as shown in Figure 11.7. Using the Distance Formula, this condition can be written as

\[
\sqrt{(x - h)^2 + (y - k)^2 + (z - j)^2} = r.
\]

By squaring each side of this equation, you obtain the standard equation of a sphere.

Standard Equation of a Sphere

The standard equation of a sphere with center \((h, k, j)\) and radius \(r\) is given by

\[
(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2.
\]

Notice the similarity of this formula to the equation of a circle in the plane.

\[
(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2 \quad \text{Equation of sphere in space}
\]

\[
(x - h)^2 + (y - k)^2 = r^2 \quad \text{Equation of circle in the plane}
\]

As is true with the equation of a circle, the equation of a sphere is simplified when the center lies at the origin. In this case, the equation is

\[
x^2 + y^2 + z^2 = r^2. \quad \text{Sphere with center at origin}
\]
Example 4  Finding the Equation of a Sphere

Find the standard equation of the sphere with center \((2, 4, 3)\) and radius 3. Does this sphere intersect the \(xy\)-plane?

Solution

\[
(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2 \quad \text{Standard equation}
\]

\[
(x - 2)^2 + (y - 4)^2 + (z - 3)^2 = 3^2 \quad \text{Substitute.}
\]

From the graph shown in Figure 11.8, you can see that the center of the sphere lies three units above the \(xy\)-plane. Because the sphere has a radius of 3, you can conclude that it does intersect the \(xy\)-plane—at the point \((2, 4, 0)\).

Example 5  Finding the Center and Radius of a Sphere

Find the center and radius of the sphere given by

\[
x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0.
\]

Solution

To obtain the standard equation of this sphere, complete the square as follows.

\[
x^2 - 2x + (\_\_\_) + (y^2 + 4y + \_\_) + (z^2 - 6z + \_\_) = -8
\]

\[
(x^2 - 2x + 1) + (y^2 + 4y + 4) + (z^2 - 6z + 9) = -8 + 1 + 4 + 9
\]

\[
(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = (\sqrt{6})^2
\]

So, the center of the sphere is \((1, -2, 3)\), and its radius is \(\sqrt{6}\). See Figure 11.9.

Note in Example 5 that the points satisfying the equation of the sphere are “surface points,” not “interior points.” In general, the collection of points satisfying an equation involving \(x\), \(y\), and \(z\) is called a **surface in space**.
Finding the intersection of a surface with one of the three coordinate planes (or with a plane parallel to one of the three coordinate planes) helps one visualize the surface. Such an intersection is called a trace of the surface. For example, the xy-trace of a surface consists of all points that are common to both the surface and the xy-plane. Similarly, the xz-trace of a surface consists of all points that are common to both the surface and the xz-plane.

**Example 6** Finding a Trace of a Surface

Sketch the xy-trace of the sphere given by \((x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 5^2\).

**Solution**

To find the xy-trace of this surface, use the fact that every point in the xy-plane has a z-coordinate of zero. By substituting \(z = 0\) into the original equation, the resulting equation will represent the intersection of the surface with the xy-plane.

\[
\begin{align*}
(x - 3)^2 + (y - 2)^2 + (z + 4)^2 &= 5^2 \\
(x - 3)^2 + (y - 2)^2 + (0 + 4)^2 &= 5^2 \\
(x - 3)^2 + (y - 2)^2 + 16 &= 25 \\
(x - 3)^2 + (y - 2)^2 &= 9 \\
(x - 3)^2 + (y - 2)^2 &= 3^2 \\
\end{align*}
\]

You can see that the xy-trace is a circle of radius 3, as shown in Figure 11.10.

**CHECK Point** Now try Exercise 71.

**TECHNOLOGY**

Most three-dimensional graphing utilities and computer algebra systems represent surfaces by sketching several traces of the surface. The traces are usually taken in equally spaced parallel planes. To graph an equation involving \(x, y,\) and \(z\) with a three-dimensional “function grapher,” you must first set the graphing mode to three-dimensional and solve the equation for \(z\). After entering the equation, you need to specify a rectangular viewing cube (the three-dimensional analog of a viewing window). For instance, to graph the top half of the sphere from Example 6, solve the equation for \(z\) to obtain the solutions \(z = -4 \pm \sqrt{25 - (x - 3)^2 - (y - 2)^2}\). The equation \(z = -4 + \sqrt{25 - (x - 3)^2 - (y - 2)^2}\) represents the top half of the sphere. Enter this equation, as shown in Figure 11.11. Next, use the viewing cube shown in Figure 11.12. Finally, you can display the graph, as shown in Figure 11.13.
11.1 Exercises

Vocabulary: Fill in the blanks.

1. A _______ coordinate system can be formed by passing a z-axis perpendicular to both the x-axis and the y-axis at the origin.
2. The three coordinate planes of a three-dimensional coordinate system are the _______ , the _______ , and the _______.
3. The coordinate planes of a three-dimensional coordinate system separate the coordinate system into eight _______.
4. The distance between the points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) can be found using the _______ _______ in Space.
5. The midpoint of the line segment joining the points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) given by the Midpoint Formula in Space is _______.
6. A _______ is the set of all points \((x, y, z)\) such that the distance between \((x, y, z)\) and a fixed point \((h, k, l)\) is \(r\).
7. A _______ in _______ is the collection of points satisfying an equation involving \(x, y,\) and \(z\).
8. The intersection of a surface with one of the three coordinate planes is called a _______ of the surface.

Skills and Applications

In Exercises 9 and 10, approximate the coordinates of the points.

9. In Exercises 11–16, plot each point in the same three-dimensional coordinate system.

11. (a) \((2, 1, 3)\)
   (b) \((-1, -1, -2)\)
12. (a) \((3, 0, 0)\)
   (b) \((-3, -2, -1)\)
13. (a) \((-3, -1, 0)\)
   (b) \((-4, 2, 2)\)
14. (a) \((0, 4, -3)\)
   (b) \((4, 0, 4)\)
15. (a) \((3, -2, 5)\)
   (b) \((-2, 4, -2)\)
16. (a) \((5, -2, 2)\)
   (b) \((-5, -2, -2)\)

In Exercises 17–20, find the coordinates of the point.

17. The point is located three units behind the \(yz\)-plane, four units to the right of the \(xz\)-plane, and five units above the \(xy\)-plane.
18. The point is located seven units in front of the \(yz\)-plane, two units to the left of the \(xz\)-plane, and one unit below the \(xy\)-plane.
19. The point is located on the \(x\)-axis, eight units in front of the \(yz\)-plane.
20. The point is located in the \(yz\)-plane, one unit to the right of the \(xz\)-plane, and six units above the \(xy\)-plane.

In Exercises 21–26, determine the octant(s) in which \((x, y, z)\) is located so that the condition(s) is (are) satisfied.

21. \(x > 0, y < 0, z > 0\)
22. \(x < 0, y > 0, z < 0\)
23. \(z > 0\)
24. \(y < 0\)
25. \(xy < 0\)
26. \(yz > 0\)

In Exercises 27–36, find the distance between the points.

27. \((0, 0, 0), (5, 2, 6)\)
28. \((1, 0, 0), (7, 0, 4)\)
29. \((3, 2, 5), (7, 4, 8)\)
30. \((4, 1, 5), (8, 2, 6)\)
31. \((-1, 4, -2), (6, 0, -9)\)
32. \((1, 1, -7), (-2, -3, -7)\)
33. \((0, -3, 0), (1, 0, -10)\)
34. \((2, -4, 0), (0, 6, -3)\)
35. \((6, -9, 1), (-2, -1, 5)\)
36. \((4, 0, -6), (8, 8, 20)\)

In Exercises 37–40, find the lengths of the sides of the right triangle with the indicated vertices. Show that these lengths satisfy the Pythagorean Theorem.

37. \((0, 0, 2), (-2, 5, 2), (0, 4, 0)\)
38. \((2, -1, 2), (-4, 4, 1), (-2, 5, 0)\)
39. \((0, 0, 0), (2, 2, 1), (2, -4, 4)\)
40. \((1, 0, 1), (1, 3, 1), (1, 0, 3)\)

In Exercises 41–44, find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.

41. \((-1, -3, -2), (5, -1, 2), (-1, 1, 2)\)
42. \((5, 3, 4), (7, 1, 3), (3, 5, 3)\)
43. \((4, -1, -2), (8, 1, 2), (2, 3, 2)\)
44. \((1, -2, -1), (3, 0, 0), (3, -6, 3)\)
In Exercises 45–52, find the midpoint of the line segment joining the points.
45. (0, 0, 0), (3, –2, 4) 46. (1, 5, –1), (2, 2, 2) 47. (3, –6, 10), (–3, 4, 4) 48. (–1, 5, –3), (3, 7, –1) 49. (–5, –2, 5), (6, 3, –7) 50. (0, –2, 5), (4, 2, 7) 51. (–2, 8, 10), (7, –4, 2) 52. (9, –5, 1), (9, –2, –4)

In Exercises 53–60, find the standard form of the equation of the sphere with the given characteristics.
53. Center: (3, 2, 4); radius: 4 54. Center: (–3, 4, 3); radius: 2 55. Center: (5, 0, –2); radius: 6 56. Center: (4, –1, 1); diameter: 5 57. Center: (–3, 7, 5); diameter: 10 58. Center: (0, 5, –9); diameter: 8 59. Endpoints of a diameter: (3, 2, 4), (0, 0, 6) 60. Endpoints of a diameter: (1, 0, 0), (5, 0, 0)

In Exercises 61–70, find the center and radius of the sphere.
61. \( x^2 + y^2 + z^2 = 6x = 0 \) 62. \( x^2 + y^2 + z^2 = 9x = 0 \) 63. \( x^2 + y^2 + z^2 = 4x + 2y = 6z + 10 = 0 \) 64. \( x^2 + y^2 + z^2 = 6x + 4y + 9 = 0 \) 65. \( x^2 + y^2 + z^2 = 4x – 8z + 19 = 0 \) 66. \( x^2 + y^2 + z^2 = 8y – 6z + 13 = 0 \) 67. \( 9x^2 + 9y^2 + 9z^2 = 18x – 6y – 7z + 73 = 0 \) 68. \( 2x^2 + 2y^2 = 2x – 6y – 4z + 5 = 0 \) 69. \( 9x^2 + 9y^2 = 9z^2 = 6x + 18y + 1 = 0 \) 70. \( 4x^2 + 4y^2 + 4z^2 = 4x – 32y + 8z + 33 = 0 \)

In Exercises 71–74, sketch the graph of the equation and sketch the specified trace.
71. \( (x – 1)^2 + y^2 + z^2 = 36; \) \( xz\)-trace 72. \( x^2 + (y + 3)^2 + z^2 = 25; \) \( yz\)-trace 73. \( (x + 2)^2 + (y – 3)^2 + z^2 = 9; \) \( yz\)-trace 74. \( x^2 + (y – 1)^2 + (z + 1)^2 = 4; \) \( xy\)-trace

In Exercises 75 and 76, use a three-dimensional graphing utility to graph the sphere.
75. \( x^2 + y^2 + z^2 = 6x – 8y – 10z + 46 = 0 \) 76. \( x^2 + y^2 + z^2 + 6y – 8z + 21 = 0 \)

**ARCHITECTURE** A spherical building has a diameter of 205 feet. The center of the building is placed at the origin of a three-dimensional coordinate system. What is the equation of the sphere?

**GEOGRAPHY** Assume that Earth is a sphere with a radius of 4000 miles. The center of Earth is placed at the origin of a three-dimensional coordinate system.
(a) What is the equation of the sphere?
(b) Lines of longitude that run north-south could be represented by what trace(s)? What shape would each of these traces form?
(c) Lines of latitude that run east-west could be represented by what trace(s)? What shape would each of these traces form?

**EXPLORATION**

**TRUE OR FALSE?** In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.

79. In the ordered triple \((x, y, z)\) that represents point \(P\) in space, \(x\) is the directed distance from the \(xy\)-plane to \(P\).
80. The surface consisting of all points \((x, y, z)\) in space that are the same distance \(r\) from the point \((h, j, k)\) has a circle as its \(xy\)-trace.

**THINK ABOUT IT** What is the \(z\)-coordinate of any point in the \(xy\)-plane? What is the \(y\)-coordinate of any point in the \(xz\)-plane? What is the \(x\)-coordinate of any point in the \(yz\)-plane?

**CAPSTONE** Find the equation of the sphere that has the points \((3, –2, 6)\) and \((-1, 4, 2)\) as endpoints of a diameter. Explain how this problem gives you a chance to use these formulas: the Distance Formula in Space, the Midpoint Formula in Space, and the standard equation of a sphere.

83. A sphere intersects the \(yz\)-plane. Describe the trace.
84. A plane intersects the \(xy\)-plane. Describe the trace.
85. A line segment has \((x_1, y_1, z_1)\) as one endpoint and \((x_m, y_m, z_m)\) as its midpoint. Find the other endpoint \((x_2, y_2, z_2)\) of the line segment in terms of \(x_1, y_1, z_1, x_m, y_m, \text{ and } z_m\).
86. Use the result of Exercise 85 to find the coordinates of the endpoint of a line segment if the coordinates of the other endpoint and the midpoint are \((3, 0, 2)\) and \((5, 8, 7)\), respectively.
11.2 VECTORS IN SPACE

What you should learn

- Find the component forms of the unit vectors in the same direction of, the magnitudes of, the dot products of, and the angles between vectors in space.
- Determine whether vectors in space are parallel or orthogonal.
- Use vectors in space to solve real-life problems.

Why you should learn it

Vectors in space can be used to represent many physical forces, such as tension in the cables used to support auditorium lights, as shown in Figure 11.14. If \( P(p_1, p_2, p_3) \) to \( Q(q_1, q_2, q_3) \), as shown in Figure 11.15, the component form of \( v \) is produced by subtracting the coordinates of the initial point from the corresponding coordinates of the terminal point

\[
v = (v_1, v_2, v_3) = (q_1 - p_1, q_2 - p_2, q_3 - p_3).
\]

Vectors in Space

Physical forces and velocities are not confined to the plane, so it is natural to extend the concept of vectors from two-dimensional space to three-dimensional space. In space, vectors are denoted by ordered triples

\[
v = (v_1, v_2, v_3).
\]

The zero vector is denoted by \( 0 = (0, 0, 0) \). Using the unit vectors \( i = (1, 0, 0) \), \( j = (0, 1, 0) \), and \( k = (0, 0, 1) \) in the direction of the positive z-axis, the standard unit vector notation for \( v \) is

\[
v = v_1i + v_2j + v_3k.
\]

as shown in Figure 11.14. If \( v \) is represented by the directed line segment from \( P(p_1, p_2, p_3) \) to \( Q(q_1, q_2, q_3) \), as shown in Figure 11.15, the component form of \( v \) is produced by subtracting the coordinates of the initial point from the corresponding coordinates of the terminal point

\[
v = (v_1, v_2, v_3) = (q_1 - p_1, q_2 - p_2, q_3 - p_3).
\]

Vectors in Space

1. Two vectors are equal if and only if their corresponding components are equal.
2. The magnitude (or length) of \( u = (u_1, u_2, u_3) \) is \( ||u|| = \sqrt{u_1^2 + u_2^2 + u_3^2} \).
3. A unit vector \( u \) in the direction of \( v \) is \( u = \frac{v}{||v||} \), \( v \neq 0 \).
4. The sum of \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) is

\[
u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3). \quad \text{Vector addition}
\]
5. The scalar multiple of the real number \( c \) and \( u = (u_1, u_2, u_3) \) is

\[
cu = (cu_1, cu_2, cu_3). \quad \text{Scalar multiplication}
\]
6. The dot product of \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) is

\[
u \cdot v = u_1v_1 + u_2v_2 + u_3v_3. \quad \text{Dot product} \]
Example 1  Finding the Component Form of a Vector

Find the component form and magnitude of the vector \( \mathbf{v} \) having initial point \((3, 4, 2)\) and terminal point \((3, 6, 4)\). Then find a unit vector in the direction of \( \mathbf{v} \).

Solution

The component form of \( \mathbf{v} \) is

\[
\mathbf{v} = (3 - 3, 6 - 4, 4 - 2) = (0, 2, 2)
\]

which implies that its magnitude is

\[
||\mathbf{v}|| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.
\]

The unit vector in the direction of \( \mathbf{v} \) is

\[
\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{1}{2\sqrt{2}} (0, 2, 2) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).
\]

Example 2  Finding the Dot Product of Two Vectors

Find the dot product of \((0, 3, -2)\) and \((4, -2, 3)\).

Solution

\[
(0, 3, -2) \cdot (4, -2, 3) = 0(4) + 3(-2) + (-2)(3)
\]

\[
= 0 - 6 - 6 = -12
\]

Note that the dot product of two vectors is a real number, not a vector.

As was discussed in Section 6.4, the angle between two nonzero vectors is the angle \( \theta \), \(0 \leq \theta \leq \pi\), between their respective standard position vectors, as shown in Figure 11.16. This angle can be found using the dot product. (Note that the angle between the zero vector and another vector is not defined.)

Angle Between Two Vectors

If \( \theta \) is the angle between two nonzero vectors \( \mathbf{u} \) and \( \mathbf{v} \), then

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||}.
\]

If the dot product of two nonzero vectors is zero, the angle between the vectors is \(90^\circ\) (recall that \(\cos 90^\circ = 0\)). Such vectors are called orthogonal. For instance, the standard unit vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) are orthogonal to each other.
Finding the Angle Between Two Vectors

Find the angle between \( \mathbf{u} = (1, 0, 2) \) and \( \mathbf{v} = (3, 1, 0) \).

**Solution**

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{(1, 0, 2) \cdot (3, 1, 0)}{\|(1, 0, 2)\| \|(3, 1, 0)\|} = \frac{3}{\sqrt{50}}
\]

This implies that the angle between the two vectors is

\[
\theta = \arccos \left(\frac{3}{\sqrt{50}}\right) = 64.9^\circ
\]

as shown in Figure 11.17.

**Example 4**  Parallel Vectors

Vector \( \mathbf{w} \) has initial point \((1, -2, 0)\) and terminal point \((3, 2, 1)\). Which of the following vectors is parallel to \( \mathbf{w} \)?

a. \( \mathbf{u} = (4, 8, 2) \)

b. \( \mathbf{v} = (4, 8, 4) \)

**Solution**

Begin by writing \( \mathbf{w} \) in component form.

\( \mathbf{w} = (3 - 1, 2 - (-2), 1 - 0) = (2, 4, 1) \)

a. Because

\[
\mathbf{u} = (4, 8, 2) \\
= 2(2, 4, 1) \\
= 2\mathbf{w}
\]

you can conclude that \( \mathbf{u} \) is parallel to \( \mathbf{w} \).

b. In this case, you need to find a scalar \( c \) such that

\[
(4, 8, 4) = c(2, 4, 1).
\]

However, equating corresponding components produces \( c = 2 \) for the first two components and \( c = 4 \) for the third. So, the equation has no solution, and the vectors \( \mathbf{v} \) and \( \mathbf{w} \) are not parallel.
You can use vectors to determine whether three points are collinear (lie on the same line). The points $P$, $Q$, and $R$ are \textit{collinear} if and only if the vectors $\overline{PQ}$ and $\overline{PR}$ are parallel.

**Example 5** Using Vectors to Determine Collinear Points

Determine whether the points $P(2, -1, 4)$, $Q(5, 4, 6)$, and $R(-4, -11, 0)$ are collinear.

**Solution**

The component forms of $\overline{PQ}$ and $\overline{PR}$ are

\[
\overline{PQ} = (5 - 2, 4 - (-1), 6 - 4) = (3, 5, 2)
\]

and

\[
\overline{PR} = (-4 - 2, -11 - (-1), 0 - 4) = (-6, -10, -4).
\]

Because $\overline{PR} = -2\overline{PQ}$, you can conclude that they are parallel. Therefore, the points $P$, $Q$, and $R$ lie on the same line, as shown in Figure 11.19.

![Figure 11.19](image)

**CHECK Point** Now try Exercise 47.

**Example 6** Finding the Terminal Point of a Vector

The initial point of the vector $\mathbf{v} = (4, 2, -1)$ is $P(3, -1, 6)$. What is the terminal point of this vector?

**Solution**

Using the component form of the vector whose initial point is $P(3, -1, 6)$ and whose terminal point is $Q(q_1, q_2, q_3)$, you can write

\[
\overline{PQ} = (q_1 - 3, q_2 + 1, q_3 - 6) = (4, 2, -1).
\]

This implies that $q_1 - 3 = 4$, $q_2 + 1 = 2$, and $q_3 - 6 = -1$. The solutions of these three equations are $q_1 = 7$, $q_2 = 1$, and $q_3 = 5$. So, the terminal point is $Q(7, 1, 5)$.

**CHECK Point** Now try Exercise 51.
Application

In Section 6.3, you saw how to use vectors to solve an equilibrium problem in a plane. The next example shows how to use vectors to solve an equilibrium problem in space.

Example 7  Solving an Equilibrium Problem

A weight of 480 pounds is supported by three ropes. As shown in Figure 11.20, the weight is located at $S(0, 2, -1)$. The ropes are tied to the points $P(2, 0, 0)$, $Q(0, 4, 0)$, and $R(-2, 0, 0)$. Find the force (or tension) on each rope.

Solution

The (downward) force of the weight is represented by the vector

$$ w = \langle 0, 0, -480 \rangle. $$

The force vectors corresponding to the ropes are as follows.

$$ u = \|u\| \frac{\overrightarrow{SP}}{\|\overrightarrow{SP}\|} = \|u\| \frac{(2 - 0, 0 - 2, 0 - (-1))}{3} = \|u\| \frac{2}{3}, \frac{-2}{3}, \frac{-1}{3} \rangle $$

$$ v = \|v\| \frac{\overrightarrow{SQ}}{\|\overrightarrow{SQ}\|} = \|v\| \frac{(0 - 0, 4 - 0, 0 - (-1))}{\sqrt{5}} = \|v\| \frac{0, 2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle $$

$$ z = \|z\| \frac{\overrightarrow{SR}}{\|\overrightarrow{SR}\|} = \|z\| \frac{(-2 - 0, 0 - 2, 0 - (-1))}{3} = \|z\| \frac{-2}{3}, \frac{-2}{3}, \frac{-1}{3} \rangle $$

For the system to be in equilibrium, it must be true that

$$ u + v + z + w = \mathbf{0} \quad \text{or} \quad u + v + z = -w. $$

This yields the following system of linear equations.

$$ \frac{2}{3}\|u\| - \frac{2}{3}\|z\| = 0 $$

$$ -\frac{2}{3}\|u\| + \frac{2}{\sqrt{5}}\|v\| - \frac{2}{3}\|z\| = 0 $$

$$ \frac{1}{3}\|u\| + \frac{1}{\sqrt{5}}\|v\| + \frac{1}{3}\|z\| = 480 $$

Using the techniques demonstrated in Chapter 7, you can find the solution of the system to be

$$ \|u\| = 360.0 $$

$$ \|v\| \approx 536.7 $$

$$ \|z\| = 360.0. $$

So, the rope attached at point $P$ has 360 pounds of tension, the rope attached at point $Q$ has about 536.7 pounds of tension, and the rope attached at point $R$ has 360 pounds of tension.

CHECK Point  Now try Exercise 59.
11.2 EXERCISES

VOCABULARY: Fill in the blanks.
1. The ______ vector is denoted by \( \mathbf{0} = (0, 0, 0) \).
2. The standard unit vector notation for a vector \( \mathbf{v} \) is given by ______.
3. The ______ ______ of a vector \( \mathbf{v} \) is produced by subtracting the coordinates of the initial point from the corresponding coordinates of the terminal point.
4. If the dot product of two nonzero vectors is zero, the angle between the vectors is \( 90^\circ \) and the vectors are called ______.
5. Two nonzero vectors \( \mathbf{u} \) and \( \mathbf{v} \) are ______ if there is some scalar such that \( \mathbf{u} = c\mathbf{v} \).
6. The points \( P, Q, \) and \( R \) are ______ if and only if the vectors \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \) are parallel.

SKILLS AND APPLICATIONS

In Exercises 7 and 8, (a) write the component form of the vector in the direction of \( \mathbf{v} \), and (b) sketch the vector with its initial point at the origin.

7. \( \mathbf{v} = 2 \mathbf{i} + 3 \mathbf{j} - 4 \mathbf{k} \)
8. \( \mathbf{v} = 1 \mathbf{i} - 2 \mathbf{j} + 3 \mathbf{k} \)

In Exercises 9 and 10, (a) write the component form of the vector \( \mathbf{v} \), (b) find the magnitude of \( \mathbf{v} \), and (c) find a unit vector in the direction of \( \mathbf{v} \).

9. Initial point: \( (-6, 4, -2) \)
   Terminal point: \( (1, -1, 3) \)
10. Initial point: \( (-7, 3, 5) \)
    Terminal point: \( (0, 0, 2) \)

In Exercises 11–14, sketch each scalar multiple of \( \mathbf{v} \).

11. \( \mathbf{v} = \langle 1, 1, 3 \rangle \)
    (a) \( 2\mathbf{v} \)  (b) \( -\mathbf{v} \)  (c) \( \frac{1}{2}\mathbf{v} \)  (d) \( 0\mathbf{v} \)
12. \( \mathbf{v} = \langle -1, 2, 2 \rangle \)
    (a) \( -\mathbf{v} \)  (b) \( 2\mathbf{v} \)  (c) \( \frac{1}{2}\mathbf{v} \)  (d) \( \frac{5}{2}\mathbf{v} \)
13. \( \mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \)
    (a) \( 2\mathbf{v} \)  (b) \( -\mathbf{v} \)  (c) \( \frac{3}{2}\mathbf{v} \)  (d) \( 0\mathbf{v} \)
14. \( \mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \)
    (a) \( 4\mathbf{v} \)  (b) \( -2\mathbf{v} \)  (c) \( \frac{1}{2}\mathbf{v} \)  (d) \( 0\mathbf{v} \)

In Exercises 15–18, find the vector \( \mathbf{z} \), given \( \mathbf{u} = \langle -1, 3, 2 \rangle \), \( \mathbf{v} = \langle 1, -2, -2 \rangle \), and \( \mathbf{w} = \langle 5, 0, -5 \rangle \).

15. \( \mathbf{z} = \mathbf{u} - 2\mathbf{v} \)
16. \( \mathbf{z} = 7\mathbf{u} + \mathbf{v} - \frac{3}{4}\mathbf{w} \)
17. \( 2\mathbf{z} - 4\mathbf{u} = \mathbf{w} \)
18. \( \mathbf{u} + \mathbf{v} + \mathbf{z} = 0 \)

In Exercises 19–28, find the magnitude of \( \mathbf{v} \).

19. \( \mathbf{v} = \langle 7, 8, 7 \rangle \)
20. \( \mathbf{v} = \langle -2, 0, -5 \rangle \)
21. \( \mathbf{v} = \langle 1, -2, 4 \rangle \)
22. \( \mathbf{v} = \langle -1, 0, 3 \rangle \)
23. \( \mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k} \)
24. \( \mathbf{v} = -\mathbf{i} - 4\mathbf{j} + 3\mathbf{k} \)
25. \( \mathbf{v} = 4\mathbf{i} - 3\mathbf{j} - 7\mathbf{k} \)
26. \( \mathbf{v} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k} \)
27. Initial point: \( (1, -3, 4) \)
   Terminal point: \( (1, 0, -1) \)
28. Initial point: \( (0, -1, 0) \)
   Terminal point: \( (1, 2, -2) \)

In Exercises 29 and 30, find a unit vector (a) in the direction of \( \mathbf{u} \) and (b) in the direction opposite of \( \mathbf{u} \).

29. \( \mathbf{u} = 8\mathbf{i} + 3\mathbf{j} - \mathbf{k} \)
30. \( \mathbf{u} = -3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k} \)

In Exercises 31–34, find the dot product of \( \mathbf{u} \) and \( \mathbf{v} \).

31. \( \mathbf{u} = \langle 4, 4, -1 \rangle \)
    \( \mathbf{v} = \langle 2, -5, -8 \rangle \)
32. \( \mathbf{u} = \langle 3, -1, 6 \rangle \)
    \( \mathbf{v} = \langle 4, -10, 1 \rangle \)
33. \( \mathbf{u} = 2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k} \)
    \( \mathbf{v} = 9\mathbf{i} + 3\mathbf{j} - \mathbf{k} \)
34. \( \mathbf{u} = 3\mathbf{j} - 6\mathbf{k} \)
    \( \mathbf{v} = 6\mathbf{i} - 4\mathbf{j} - 2\mathbf{k} \)
In Exercises 35–38, find the angle $\theta$ between the vectors.

35. $\mathbf{u} = \langle 0, 2, 2 \rangle$ \hspace{1cm} 36. $\mathbf{u} = \langle -1, 3, 0 \rangle$
   $\mathbf{v} = \langle 3, 0, -4 \rangle$ \hspace{1cm} $\mathbf{v} = \langle 1, 2, -1 \rangle$
37. $\mathbf{u} = 10\mathbf{i} + 40\mathbf{j}$ \hspace{1cm} 38. $\mathbf{u} = 8\mathbf{j} - 20\mathbf{k}$
   $\mathbf{v} = -3\mathbf{j} + 8\mathbf{k}$ \hspace{1cm} $\mathbf{v} = 10\mathbf{i} - 5\mathbf{k}$

In Exercises 39–46, determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, parallel, or neither.

39. $\mathbf{u} = \langle -12, 6, 15 \rangle$ \hspace{1cm} 40. $\mathbf{u} = \langle -1, 3, -1 \rangle$
   $\mathbf{v} = \langle 8, -4, -10 \rangle$ \hspace{1cm} $\mathbf{v} = \langle 2, -1, 5 \rangle$
41. $\mathbf{u} = \langle 0, 1, 6 \rangle$ \hspace{1cm} 42. $\mathbf{u} = \langle 0, 4, -1 \rangle$
   $\mathbf{v} = \langle 1, -2, -1 \rangle$ \hspace{1cm} $\mathbf{v} = \langle 1, 0, 0 \rangle$
43. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + 2\mathbf{k}$ \hspace{1cm} 44. $\mathbf{u} = -\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$
   $\mathbf{v} = 4\mathbf{i} + 10\mathbf{j} + \mathbf{k}$ \hspace{1cm} $\mathbf{v} = 8\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$
45. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ \hspace{1cm} 46. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
   $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ \hspace{1cm} $\mathbf{v} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$

In Exercises 47–50, use vectors to determine whether the points are collinear.

47. $(5, 4, 1), (7, 3, -1), (4, 5, 3)$
48. $(-2, 7, 4), (-4, 8, 1), (0, 6, 7)$
49. $(1, 3, 2), (-1, 2, 5), (3, 4, -1)$
50. $(0, 4, 4), (-1, 5, 6), (-2, 6, 7)$

In Exercises 51–54, the vector $\mathbf{v}$ and its initial point are given. Find the terminal point.

51. $\mathbf{v} = \langle 2, -4, 7 \rangle$
   Initial point: $(1, 5, 0)$
52. $\mathbf{v} = \langle 4, -1, -1 \rangle$
   Initial point: $(6, -4, 3)$
53. $\mathbf{v} = \langle 4, \frac{1}{2}, -\frac{1}{2} \rangle$
   Initial point: $(2, 1, -\frac{1}{2})$
54. $\mathbf{v} = \langle 2, -\frac{1}{2}, 4 \rangle$
   Initial point: $(3, 2, -\frac{1}{2})$

55. Determine the values of $c$ such that $\|c\mathbf{u}\| = 3$, where $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
56. Determine the values of $c$ such that $\|c\mathbf{u}\| = 12$, where $\mathbf{u} = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

In Exercises 57 and 58, write the component form of $\mathbf{v}$.

57. $\mathbf{v}$ lies in the $yz$-plane, has magnitude 4, and makes an angle of $45^\circ$ with the positive $y$-axis.
58. $\mathbf{v}$ lies in the $xz$-plane, has magnitude 10, and makes an angle of $60^\circ$ with the positive $z$-axis.

59. TENSION The weight of a crate is 500 newtons. Find the tension in each of the supporting cables shown in the figure.

![Figure for Exercise 59](image1.png)

60. TENSION The lights in an auditorium are 24-pound disks of radius 18 inches. Each disk is supported by three equally spaced cables that are $L$ inches long (see figure).

(a) Write the tension $T$ in each cable as a function of $L$. Determine the domain of the function.

(b) Use the function from part (a) to complete the table.

<table>
<thead>
<tr>
<th>$L$</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Use a graphing utility to graph the function in part (a). What are the asymptotes of the graph? Interpret their meaning in the context of the problem.

(d) Determine the minimum length of each cable if a cable can carry a maximum load of 10 pounds.

EXPLORATION

TRUE OR FALSE? In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.

61. If the dot product of two nonzero vectors is zero, then the angle between the vectors is a right angle.
62. If $AB$ and $AC$ are parallel vectors, then points $A$, $B$, and $C$ are collinear.

63. What is known about the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ if $\mathbf{u} \cdot \mathbf{v} < 0$? Explain.

64. CAPSTONE Consider the two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. Describe the geometric figure generated by the terminal points of the vectors $s\mathbf{v} + t\mathbf{u}$, $\mathbf{u} + \mathbf{v}$, and $s\mathbf{u} + t\mathbf{v}$ where $s$ and $t$ represent real numbers.
The Cross Product

Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors. In this section, you will study a product that will yield such a vector. It is called the cross product, and it is conveniently defined and calculated using the standard unit vector form.

It is important to note that this definition applies only to three-dimensional vectors. The cross product is not defined for two-dimensional vectors.

A convenient way to calculate \( \mathbf{u} \times \mathbf{v} \) is to use the following determinant form with cofactor expansion. (This determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because the entries of the corresponding matrix are not all real numbers.)

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
u_2 & u_3 & v_3 \\
v_2 & v_3 & v_1
\end{vmatrix} = (u_2v_3 - u_3v_2)i - (u_1v_3 - u_3v_1)j + (u_1v_2 - u_2v_1)k
\]

Note the minus sign in front of the \( j \)-component. Recall from Section 8.4 that each of the three \( 2 \times 2 \) determinants can be evaluated by using the following pattern.

\[
\begin{vmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{vmatrix} = a_1b_2 - a_2b_1
\]
Example 1 Finding Cross Products

Given \( \mathbf{u} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} \) and \( \mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} \), find each cross product.

a. \( \mathbf{u} \times \mathbf{v} \)  b. \( \mathbf{v} \times \mathbf{u} \)  c. \( \mathbf{v} \times \mathbf{v} \)

Solution

a. \( \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k} = (4 - 1)\mathbf{i} - (2 - 3)\mathbf{j} + (1 - 6)\mathbf{k} = 3\mathbf{i} + \mathbf{j} - 5\mathbf{k} \\

b. \( \mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k} = (1 - 4)\mathbf{i} - (3 - 2)\mathbf{j} + (1 - 6)\mathbf{k} = -3\mathbf{i} - \mathbf{j} + 5\mathbf{k} \\

c. \( \mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix} = 0 \\

CHECK POINT Now try Exercise 25.

The results obtained in Example 1 suggest some interesting algebraic properties of the cross product. For instance,

\[ \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \quad \text{and} \quad \mathbf{v} \times \mathbf{v} = 0. \]

These properties, and several others, are summarized in the following list.

**Algebraic Properties of the Cross Product**

Let \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) be vectors in space and let \( c \) be a scalar.

1. \( \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \)
2. \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \)
3. \( c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) \)
4. \( \mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0} \)
5. \( \mathbf{u} \times \mathbf{u} = \mathbf{0} \)
6. \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \)

For proofs of the Algebraic Properties of the Cross Product, see Proofs in Mathematics on page 845.
Geometric Properties of the Cross Product

The first property listed on the preceding page indicates that the cross product is *not* commutative. In particular, this property indicates that the vectors \( \mathbf{u} \) and \( \mathbf{v} \) have equal lengths but opposite directions. The following list gives some other geometric properties of the cross product of two vectors.

For proofs of the Geometric Properties of the Cross Product, see Proofs in Mathematics on page 846.

Both \( \mathbf{u} \times \mathbf{v} \) and \( \mathbf{v} \times \mathbf{u} \) are perpendicular to the plane determined by \( \mathbf{u} \) and \( \mathbf{v} \). One way to remember the orientations of the vectors \( \mathbf{u}, \mathbf{v} \), and \( \mathbf{k} = \mathbf{i} \times \mathbf{j} \) as shown in Figure 11.21. The three vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{k} \) form a *right-handed system*.

**Example 2** Using the Cross Product

Find a unit vector that is orthogonal to both

\[
\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{v} = -3\mathbf{i} + 6\mathbf{j}.
\]

**Solution**

The cross product \( \mathbf{u} \times \mathbf{v} \), as shown in Figure 11.22, is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
3 & -4 & 1 \\
-3 & 6 & 0
\end{vmatrix}
= -6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}
\]

Because

\[
\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-6)^2 + (-3)^2 + 6^2}
= \sqrt{81}
= 9
\]

a unit vector orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \) is

\[
\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{2}{3} \mathbf{i} - \frac{1}{3} \mathbf{j} + \frac{2}{3} \mathbf{k}.
\]

**Check Point** Now try Exercise 31.
Section 11.3  The Cross Product of Two Vectors

In Example 2, note that you could have used the cross product \( \mathbf{v} \times \mathbf{u} \) to form a unit vector that is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \). With that choice, you would have obtained the negative of the unit vector found in the example.

The fourth geometric property of the cross product states that \( \| \mathbf{u} \times \mathbf{v} \| \) is the area of the parallelogram that has \( \mathbf{u} \) and \( \mathbf{v} \) as adjacent sides. A simple example of this is given by the unit square with adjacent sides of \( \mathbf{i} \) and \( \mathbf{j} \). Because

\[
\mathbf{i} \times \mathbf{j} = \mathbf{k}
\]

and \( \| \mathbf{k} \| = 1 \), it follows that the square has an area of 1. This geometric property of the cross product is illustrated further in the next example.

**Example 3**  Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram. Then find the area of the parallelogram. Is the parallelogram a rectangle?

\[A(5, 2, 0), \quad B(2, 6, 1), \quad C(2, 4, 7), \quad D(5, 0, 6)\]

**Solution**

From Figure 11.23 you can see that the sides of the quadrilateral correspond to the following four vectors.

\[
\begin{align*}
\overrightarrow{AB} &= -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} \\
\overrightarrow{CD} &= 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB} \\
\overrightarrow{AD} &= 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} \\
\overrightarrow{CB} &= 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD}
\end{align*}
\]

Because \( \overrightarrow{CD} = -\overrightarrow{AB} \) and \( \overrightarrow{CB} = -\overrightarrow{AD} \), you can conclude that \( \overrightarrow{AB} \) is parallel to \( \overrightarrow{CD} \) and \( \overrightarrow{AD} \) is parallel to \( \overrightarrow{CB} \). It follows that the quadrilateral is a parallelogram with \( \overrightarrow{AB} \) and \( \overrightarrow{AD} \) as adjacent sides. Moreover, because

\[
\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-3 & 4 & 1 \\
0 & -2 & 6
\end{vmatrix} = 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}
\]

the area of the parallelogram is

\[
\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{26^2 + 18^2 + 6^2} = \sqrt{1036} \approx 32.19.
\]

You can tell whether the parallelogram is a rectangle by finding the angle between the vectors \( \overrightarrow{AB} \) and \( \overrightarrow{AD} \).

\[
\sin \theta = \frac{\|\overrightarrow{AB} \times \overrightarrow{AD}\|}{\|\overrightarrow{AB}\| \|\overrightarrow{AD}\|} = \frac{\sqrt{1036}}{26\sqrt{40}}
\]

\[
\sin \theta \approx 0.998 \\
\theta \approx \arcsin 0.998 \\
\theta \approx 86.4^\circ
\]

Because \( \theta \neq 90^\circ \), the parallelogram is not a rectangle.

**CHECK**  Now try Exercise 43.
The Triple Scalar Product

For the vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) in space, the dot product of \( \mathbf{u} \) and \( \mathbf{v} \times \mathbf{w} \) is called the triple scalar product of \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w}. \)

### The Triple Scalar Product

For \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}, \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}, \) and \( \mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}, \) the **triple scalar product** is given by

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix}
    u_1 & u_2 & u_3 \\
    v_1 & v_2 & v_3 \\
    w_1 & w_2 & w_3
\end{vmatrix}.
\]

If the vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) do not lie in the same plane, the triple scalar product \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \) can be used to determine the volume of the parallelepiped (a polyhedron, all of whose faces are parallelograms) with \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) as adjacent edges, as shown in Figure 11.24.

### Geometric Property of the Triple Scalar Product

The volume \( V \) of a parallelepiped with vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) as adjacent edges is given by

\[
V = \left| \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \right|.
\]

#### Example 4 Volume by the Triple Scalar Product

Find the volume of the parallelepiped having

\[
\mathbf{u} = 3 \mathbf{i} - 5 \mathbf{j} + \mathbf{k}, \quad \mathbf{v} = 2 \mathbf{j} - 2 \mathbf{k}, \quad \text{and} \quad \mathbf{w} = 3 \mathbf{i} + \mathbf{j} + \mathbf{k}
\]

as adjacent edges, as shown in Figure 11.25.

**Solution**

The value of the triple scalar product is

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix}
    3 & -5 & 1 \\
    0 & 2 & -2 \\
    3 & 1 & 1
\end{vmatrix} = 3 \begin{vmatrix}
    2 & -2 \\
    1 & 1
\end{vmatrix} - (-5) \begin{vmatrix}
    0 & -2 \\
    3 & 1
\end{vmatrix} + 1 \begin{vmatrix}
    0 & 2 \\
    3 & 1
\end{vmatrix} = 3(4) + 5(6) + 1(-6) = 36.
\]

So, the volume of the parallelepiped is

\[
| \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) | = 36 = 36.
\]

**Check Point** Now try Exercise 57.
VOCABULARY: Fill in the blanks.

1. To find a vector in space that is orthogonal to two given vectors, find the ________ ________ of the two vectors.
2. \( \mathbf{u} \times \mathbf{u} = ________ \\
3. \( \| \mathbf{u} \times \mathbf{v} \| = ________ \\
4. The dot product of \( \mathbf{u} \) and \( \mathbf{v} \times \mathbf{w} \) is called the ________ ________ ________ of \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \).

SKILLS AND APPLICATIONS

In Exercises 5–10, find the cross product of the unit vectors and sketch the result.

5. \( \mathbf{j} \times \mathbf{i} \)  
6. \( \mathbf{i} \times \mathbf{j} \)  
7. \( \mathbf{i} \times \mathbf{k} \)  
8. \( \mathbf{k} \times \mathbf{i} \)  
9. \( \mathbf{j} \times \mathbf{k} \)  
10. \( \mathbf{k} \times \mathbf{j} \)

In Exercises 11–20, use the vectors \( \mathbf{u} \) and \( \mathbf{v} \) to find each expression.

11. \( \mathbf{u} \times \mathbf{v} \)  
12. \( \mathbf{v} \times \mathbf{u} \)  
13. \( \mathbf{v} \times \mathbf{v} \)  
14. \( \mathbf{v} \times ( \mathbf{u} \times \mathbf{u} ) \)  
15. \((3\mathbf{u}) \times \mathbf{v}\)  
16. \( \mathbf{u} \times (2\mathbf{v}) \)  
17. \( \mathbf{u} \times (-\mathbf{v}) \)  
18. \((-2\mathbf{u}) \times \mathbf{v}\)  
19. \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) \)  
20. \( \mathbf{v} \cdot (\mathbf{v} \times \mathbf{u}) \)

In Exercises 21–30, find \( \mathbf{u} \times \mathbf{v} \) and show that it is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).

21. \( \mathbf{u} = (2, -3, 4) \)  
22. \( \mathbf{u} = (6, 8, 3) \)  
23. \( \mathbf{u} = (-10, 0, 6) \)  
24. \( \mathbf{u} = (7, 0, 0) \)  
25. \( \mathbf{u} = 6\mathbf{i} + 2\mathbf{j} + \mathbf{k} \)  
26. \( \mathbf{u} = 1 + \frac{1}{2}\mathbf{j} - \frac{5}{2}\mathbf{k} \)  
27. \( \mathbf{u} = 6\mathbf{k} \)  
28. \( \mathbf{u} = \frac{3}{2}\mathbf{j} \)  
29. \( \mathbf{u} = -i + 3\mathbf{j} + \mathbf{k} \)  
30. \( \mathbf{u} = -i + \mathbf{k} \)

In Exercises 31–36, find a unit vector orthogonal to \( \mathbf{u} \) and \( \mathbf{v} \).

31. \( \mathbf{u} = 3\mathbf{i} + \mathbf{j} \)  
32. \( \mathbf{u} = \mathbf{i} + 2\mathbf{j} \)  
33. \( \mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \)  
34. \( \mathbf{u} = 7\mathbf{i} - 14\mathbf{j} + 5\mathbf{k} \)  
35. \( \mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k} \)  
36. \( \mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \)

In Exercises 37–42, find the area of the parallelogram that has the vectors as adjacent sides.

37. \( \mathbf{u} = \mathbf{k} \)  
38. \( \mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \)  
39. \( \mathbf{u} = 4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k} \)  
40. \( \mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \)

In Exercises 43–46, (a) verify that the points are the vertices of a parallelogram, (b) find its area, and (c) determine whether the parallelogram is a rectangle.

43. \( \mathbf{A} = (2, -1, 4), \mathbf{B} = (3, 1, 2), \mathbf{C} = (0, 5, 6), \mathbf{D} = (-1, 3, 8) \)
44. \( \mathbf{A} = (1, 1, 1), \mathbf{B} = (2, 3, 4), \mathbf{C} = (6, 5, 2), \mathbf{D} = (7, 7, 5) \)
45. \( \mathbf{A} = (3, 2, -1), \mathbf{B} = (-2, 2, -3), \mathbf{C} = (3, 5, -2), \mathbf{D} = (-2, 5, -4) \)
46. \( \mathbf{A} = (2, 1, 1), \mathbf{B} = (2, 3, 1), \mathbf{C} = (-2, 4, 1), \mathbf{D} = (-2, 6, 1) \)

In Exercises 47–50, find the area of the triangle with the given vertices. (The area \( A \) of the triangle having \( \mathbf{u} \) and \( \mathbf{v} \) as adjacent sides is given by \( A = \frac{1}{2} |\mathbf{u} \times \mathbf{v}| \).)

47. \( \mathbf{u} = (0, 0, 0), (1, 2, 3), (-3, 0, 0) \)
48. \( \mathbf{u} = (1, -4, 3), (2, 0, 2), (-2, 2, 0) \)
49. \( \mathbf{u} = (2, 3, -5), (-2, -2, 0), (3, 0, 6) \)
50. \( \mathbf{u} = (2, 4, 0), (-2, -4, 0), (0, 0, 4) \)

In Exercises 51–54, find the triple scalar product.

51. \( \mathbf{u} = (3, 4, 4), \mathbf{v} = (2, 3, 0), \mathbf{w} = (0, 0, 6) \)
52. \( \mathbf{u} = (4, 0, 1), \mathbf{v} = (0, 5, 0), \mathbf{w} = (0, 0, 1) \)
53. \( \mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \mathbf{v} = \mathbf{i} - \mathbf{j}, \mathbf{w} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k} \)
54. \( \mathbf{u} = 4\mathbf{j} - 7\mathbf{k}, \mathbf{v} = 2\mathbf{i} + 4\mathbf{k}, \mathbf{w} = -3\mathbf{j} + 6\mathbf{k} \)
In Exercises 55–58, use the triple scalar product to find the volume of the parallelepiped having adjacent edges \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \).

55. \( \mathbf{u} = \mathbf{i} + \mathbf{j} \)  
   \( \mathbf{v} = \mathbf{j} + \mathbf{k} \)  
   \( \mathbf{w} = \mathbf{i} + \mathbf{k} \)

56. \( \mathbf{u} = \mathbf{i} + \mathbf{j} + 3\mathbf{k} \)  
   \( \mathbf{v} = 3\mathbf{j} + 3\mathbf{k} \)  
   \( \mathbf{w} = 3\mathbf{i} + 3\mathbf{k} \)

57. \( \mathbf{u} = (0, 2, 2) \)  
   \( \mathbf{v} = (0, 0, -2) \)  
   \( \mathbf{w} = (3, 0, 2) \)

58. \( \mathbf{u} = (1, 2, -1) \)  
   \( \mathbf{v} = (-1, 2, 2) \)  
   \( \mathbf{w} = (2, 0, 1) \)

In Exercises 59 and 60, find the volume of the parallelepiped with the given vertices.

59. \( A(0, 0, 0), B(4, 0, 0), C(4, -2, 3), D(0, -2, 3), E(4, 5, 3), F(0, 5, 3), G(0, 3, 6), H(4, 3, 6) \)

60. \( A(0, 0, 0), B(1, 1, 0), C(1, 0, 2), D(0, 1, 1), E(2, 1, 2), F(1, 1, 3), G(1, 2, 1), H(2, 2, 3) \)

61. **TORQUE** The brakes on a bicycle are applied by using a downward force of \( p \) pounds on the pedal when the six-inch crank makes a 40° angle with the horizontal (see figure). Vectors representing the position of the crank and the force are \( \mathbf{v} = \frac{1}{2}(-\cos 40^\circ \mathbf{j} - \sin 40^\circ \mathbf{k}) \) and \( \mathbf{F} = -p\mathbf{k} \), respectively.

(a) The magnitude of the torque on the crank is given by \( |\mathbf{v} \times \mathbf{F}| \). Using the given information, write the torque \( T \) on the crank as a function of \( p \).

(b) Use the function from part (a) to complete the table.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>

62. **TORQUE** Both the magnitude and direction of the force on a crankshaft change as the crankshaft rotates. Use the technique given in Exercise 61 to find the magnitude of the torque on the crankshaft using the position and data shown in the figure.

**EXPLORATION**

**TRUE OR FALSE?** In Exercises 63 and 64, determine whether the statement is true or false. Justify your answer.

63. The cross product is not defined for vectors in the plane.

64. If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in space that are nonzero and nonparallel, then \( \mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u} \).

65. **THINK ABOUT IT** Calculate \( \mathbf{u} \times \mathbf{v} \) and \( -(\mathbf{v} \times \mathbf{u}) \) for several values of \( \mathbf{u} \) and \( \mathbf{v} \). What do your results imply? Interpret your results geometrically.

66. **THINK ABOUT IT** If the magnitudes of two vectors are doubled, how will the magnitude of the cross product of the vectors change?

67. **THINK ABOUT IT** If you connect the terminal points of two vectors \( \mathbf{u} \) and \( \mathbf{v} \) that have the same initial points, a triangle is formed. Is it possible to use the cross product \( \mathbf{u} \times \mathbf{v} \) to determine the area of the triangle? Explain. Verify your conclusion using two vectors from Example 3.

68. **CAPSTONE** Define the cross product of two vectors in space, \( \mathbf{u} \) and \( \mathbf{v} \), where \( \mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \) and \( \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \). Explain, in your own words, what the cross product \( \mathbf{u} \times \mathbf{v} \) represents. What does it mean when \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \)?

69. **PROOF** Consider the vectors \( \mathbf{u} = (\cos \alpha, \sin \alpha, 0) \) and \( \mathbf{v} = (\cos \beta, \sin \beta, 0) \), where \( \alpha > \beta \). Find the cross product of the vectors and use the result to prove the identity \( \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \).
Lines and Planes in Space

What you should learn
• Find parametric and symmetric equations of lines in space.
• Find equations of planes in space.
• Sketch planes in space.
• Find distances between points and planes in space.

Why you should learn it
Equations in three variables can be used to model real-life data. For instance, in Exercise 61 on page 839, you will determine how changes in the consumption of two types of beverages affect the consumption of a third type of beverage.

Lines in Space
In the plane, slope is used to determine an equation of a line. In space, it is more convenient to use vectors to determine the equation of a line.

In Figure 11.26, consider the line through the point \( P(x_1, y_1, z_1) \) and parallel to the vector \( \mathbf{v} = \langle a, b, c \rangle \).

The vector \( \mathbf{v} \) is the direction vector for the line \( L \), and \( a \), \( b \), and \( c \) are the direction numbers. One way of describing the line \( L \) is to say that it consists of all points \( Q(x, y, z) \) for which the vector \( \overrightarrow{PQ} \) is parallel to \( \mathbf{v} \). This means that \( \overrightarrow{PQ} \) is a scalar multiple of \( \mathbf{v} \), and you can write \( \overrightarrow{PQ} = t \mathbf{v} \), where \( t \) is a scalar.

\[
\overrightarrow{PQ} = \langle x - x_1, y - y_1, z - z_1 \rangle = \langle at, bt, ct \rangle = t \mathbf{v}
\]

By equating corresponding components, you can obtain the parametric equations of a line in space.

Parametric Equations of a Line in Space
A line \( L \) parallel to the vector \( \mathbf{v} = \langle a, b, c \rangle \) and passing through the point \( P(x_1, y_1, z_1) \) is represented by the parametric equations

\[
x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.
\]

If the direction numbers \( a \), \( b \), and \( c \) are all nonzero, you can eliminate the parameter \( t \) to obtain the symmetric equations of a line.

\[
\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \text{Symmetric equations}
\]
Finding Parametric and Symmetric Equations

Find parametric and symmetric equations of the line that passes through the point \((1, -2, 4)\) and is parallel to \(v = \langle 2, 4, -4 \rangle\).

Solution

To find a set of parametric equations of the line, use the coordinates \(x_1 = 1, y_1 = -2, z_1 = 4\) and direction numbers \(a = 2, b = 4, c = -4\) (see Figure 11.27).

\[
\begin{align*}
x &= 1 + 2t, \\
y &= -2 + 4t, \\
z &= 4 - 4t
\end{align*}
\]

Parametric equations

Because \(a, b, c\) are all nonzero, a set of symmetric equations is

\[
\frac{x - 1}{2} = \frac{y + 2}{4} = \frac{z - 4}{-4}.
\]

Symmetric equations

**CHECK Point** Now try Exercise 5.

Neither the parametric equations nor the symmetric equations of a given line are unique. For instance, in Example 1, by letting \(t = 1\) in the parametric equations you would obtain the point \((3, 2, 0)\). Using this point with the direction numbers \(a = 2, b = 4, c = -4\) produces the parametric equations

\[
\begin{align*}
x &= 3 + 2t, \\
y &= 2 + 4t, \\
z &= -4t
\end{align*}
\]

Example 2  
Parametric and Symmetric Equations of a Line Through Two Points

Find a set of parametric and symmetric equations of the line that passes through the points \((-2, 1, 0)\) and \((1, 3, 5)\).

Solution

Begin by letting \(P = (-2, 1, 0)\) and \(Q = (1, 3, 5)\). Then a direction vector for the line passing through \(P\) and \(Q\) is

\[
v = PQ = \langle 1 - (-2), 3 - 1, 5 - 0 \rangle = \langle 3, 2, 5 \rangle.
\]

Using the direction numbers \(a = 3, b = 2, c = 5\) with the initial point \(P(-2, 1, 0)\), you can obtain the parametric equations

\[
\begin{align*}
x &= -2 + 3t, \\
y &= 1 + 2t, \\
z &= 5t
\end{align*}
\]

Parametric equations

Because \(a, b, c\) are all nonzero, a set of symmetric equations is

\[
\begin{align*}
\frac{x + 2}{3} &= \frac{y - 1}{2} = \frac{z}{5}
\end{align*}
\]

Symmetric equations

**CHECK Point** Now try Exercise 11.
Planes in Space

You have seen how an equation of a line in space can be obtained from a point on the line and a vector parallel to it. You will now see that an equation of a plane in space can be obtained from a point in the plane and a vector normal (perpendicular) to the plane.

Consider the plane containing the point having a nonzero normal vector as shown in Figure 11.28. This plane consists of all points for which the vector is orthogonal to . Using the dot product, you can write

\[ \mathbf{n} \cdot \overrightarrow{PQ} = 0 \]

The third equation of the plane is said to be in standard form.

Regrouping terms yields the general form of the equation of a plane in space

\[ ax + by + cz + d = 0. \]

Given the general form of the equation of a plane, it is easy to find a normal vector to the plane. Use the coefficients of , , and to write \( \mathbf{n} = (a, b, c). \)
Finding an Equation of a Plane in Three-Space

Find the general form of the equation of the plane passing through the points \((2, 1, 1), (0, 4, 1),\) and \((-2, 1, 4).\)

**Solution**

To find the equation of the plane, you need a point in the plane and a vector that is normal to the plane. There are three choices for the point, but no normal vector is given. To obtain a normal vector, use the cross product of vectors extending from the point \((2, 1, 1)\) to the points \((0, 4, 1)\) and \((-2, 1, 4)\), as shown in Figure 11.29. The component forms of \(\mathbf{u}\) and \(\mathbf{v}\) are

\[
\mathbf{u} = \langle 0 - 2, 4 - 1, 1 - 1 \rangle = \langle -2, 3, 0 \rangle \\
\mathbf{v} = \langle -2 - 2, 1 - 1, 4 - 1 \rangle = \langle -4, 0, 3 \rangle
\]

and it follows that

\[
\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-2 & 3 & 0 \\
-4 & 0 & 3
\end{vmatrix} = 9\mathbf{i} + 6\mathbf{j} + 12\mathbf{k} = \langle a, b, c \rangle
\]

is normal to the given plane. Using the direction numbers for \(\mathbf{n}\) and the initial point \((x_1, y_1, z_1) = (2, 1, 1),\) you can determine an equation of the plane to be

\[
a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \\
9(x - 2) + 6(y - 1) + 12(z - 1) = 0 \quad \text{Standard form} \\
9x + 6y + 12z - 36 = 0 \\
3x + 2y + 4z - 12 = 0. \quad \text{General form}
\]

Check that each of the three points satisfies the equation \(3x + 2y + 4z - 12 = 0.\)

**Check Point**

Now try Exercise 29.

Two distinct planes in three-space either are parallel or intersect in a line. If they intersect, you can determine the angle \(\theta (0 \leq \theta \leq 90^\circ)\) between them from the angle between their normal vectors, as shown in Figure 11.30. Specifically, if vectors \(\mathbf{n}_1\) and \(\mathbf{n}_2\) are normal to two intersecting planes, the angle \(\theta\) between the normal vectors is equal to the **angle between the two planes** and is given by

\[
\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}
\]

**Angle between two planes**

Consequently, two planes with normal vectors \(\mathbf{n}_1\) and \(\mathbf{n}_2\) are

1. **perpendicular** if \(\mathbf{n}_1 \cdot \mathbf{n}_2 = 0.\)
2. **parallel** if \(\mathbf{n}_1\) is a scalar multiple of \(\mathbf{n}_2.\)
Finding the Line of Intersection of Two Planes

Find the angle between the two planes given by

Equation for plane 1
\[ x - 2y + z = 0 \]

Equation for plane 2
\[ 2x + 3y - 2z = 0 \]

and find parametric equations of their line of intersection (see Figure 11.31).

Solution

The normal vectors for the planes are \( \mathbf{n}_1 = (1, -2, 1) \) and \( \mathbf{n}_2 = (2, 3, -2) \). Consequently, the angle between the two planes is determined as follows.

\[
\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|-6|}{\sqrt{6} \sqrt{17}} = \frac{6}{\sqrt{102}} \approx 0.59409.
\]

This implies that the angle between the two planes is \( \theta \approx 53.55^\circ \). You can find the line of intersection of the two planes by simultaneously solving the two linear equations representing the planes. One way to do this is to multiply the first equation by \(-2\) and add the result to the second equation.

\[
\begin{align*}
7y - 4z &= 0 \\
7y &= 4z
\end{align*}
\]

Substituting \( y = \frac{4z}{7} \) back into one of the original equations, you can determine that \( x = \frac{z}{7} \). Finally, by letting \( t = \frac{z}{7} \), you obtain the parametric equations

\[
\begin{align*}
x &= t = x_1 + at & \text{Parametric equation for } x \\
y &= 4t = y_1 + bt & \text{Parametric equation for } y \\
z &= 7t = z_1 + ct. & \text{Parametric equation for } z
\end{align*}
\]

Because \((x_1, y_1, z_1) = (0, 0, 0)\) lies in both planes, you can substitute for \(x_1, y_1, \) and \(z_1\) in these parametric equations, which indicates that \(a = 1, b = 4,\) and \(c = 7\) are direction numbers for the line of intersection.

CHECK Point Now try Exercise 47.

Note that the direction numbers in Example 4 can also be obtained from the cross product of the two normal vectors as follows.

\[
\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix}
i & j & k \\
1 & -2 & 1 \\
2 & 3 & -2
\end{vmatrix} = \begin{vmatrix}1 & 1 \\
1 & 2
\end{vmatrix} k = 3k
\]

This means that the line of intersection of the two planes is parallel to the cross product of their normal vectors.
Sketching Planes in Space

As discussed in Section 11.1, if a plane in space intersects one of the coordinate planes, the line of intersection is called the trace of the given plane in the coordinate plane. To sketch a plane in space, it is helpful to find its points of intersection with the coordinate axes and its traces in the coordinate planes. For example, consider the plane

\[ 3x + 2y + 4z = 12. \]

Equation of plane

You can find the \( xy \)-trace by letting \( z = 0 \) and sketching the line

\[ 3x + 2y = 12 \]

\( xy \)-trace

in the \( xy \)-plane. This line intersects the \( x \)-axis at \((4, 0, 0)\) and the \( y \)-axis at \((0, 6, 0)\). In Figure 11.32, this process is continued by finding the \( yz \)-trace and the \( xz \)-trace and then shading the triangular region lying in the first octant.

If the equation of a plane has a missing variable, such as \( 2x + z = 1 \), the plane must be parallel to the \( x \)-axis represented by the missing variable, as shown in Figure 11.33. If two variables are missing from the equation of a plane, then it is parallel to the coordinate plane represented by the missing variables, as shown in Figure 11.34.
Distance Between a Point and a Plane

The distance between a point \( Q \) and a plane is the length of the shortest line segment connecting \( Q \) to the plane, as shown in Figure 11.35. If \( P \) is any point in the plane, you can find this distance by projecting the vector \( \overrightarrow{PQ} \) onto the normal vector \( \mathbf{n} \). The length of this projection is the desired distance.

Distance Between a Point and a Plane

The distance between a plane and a point \( Q \) (not in the plane) is

\[
D = \left\| \text{proj}_n \overrightarrow{PQ} \right\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{||\mathbf{n}||}
\]

where \( P \) is a point in the plane and \( \mathbf{n} \) is normal to the plane.

To find a point in the plane given by \( ax + by + cz + d = 0 \), where \( a \neq 0 \), let \( y = 0 \) and \( z = 0 \). Then, from the equation \( ax + d = 0 \), you can conclude that the point \((-d/a, 0, 0)\) lies in the plane.

Example 5 Finding the Distance Between a Point and a Plane

Find the distance between the point \( Q(1, 5, -4) \) and the plane \( 3x - y + 2z = 6 \).

Solution

You know that \( \mathbf{n} = (3, -1, 2) \) is normal to the given plane. To find a point in the plane, let \( y = 0 \) and \( z = 0 \), and obtain the point \( P(2, 0, 0) \). The vector from \( P \) to \( Q \) is

\[
\overrightarrow{PQ} = (1 - 2, 5 - 0, -4 - 0) = (-1, 5, -4).
\]

The formula for the distance between a point and a plane produces

\[
D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{||\mathbf{n}||} = \frac{|(-1, 5, -4) \cdot (3, -1, 2)|}{\sqrt{9 + 1 + 4}} = \frac{|-3 - 5 - 8|}{\sqrt{14}} = \frac{16}{\sqrt{14}}
\]

CHECK POINT Now try Exercise 59.

The choice of the point \( P \) in Example 5 is arbitrary. Try choosing a different point to verify that you obtain the same distance.

11.4 EXERCISES

VOCABULARY: Fill in the blanks.

1. The ______ vector for a line L is given by \( v = \) ______.
2. The ______ ______ of a line in space are given by \( x = x_1 + at, y = y_1 + bt, \) and \( z = z_1 + ct. \)
3. If the direction numbers \( a, b, \) and \( c \) of the vector \( v = (a, b, c) \) are all nonzero, you can eliminate the parameter to obtain the ______ ______ of a line.
4. A vector that is perpendicular to a plane is called ______.

SKILLS AND APPLICATIONS

In Exercises 5–10, find a set of (a) parametric equations and (b) symmetric equations for the line through the point and parallel to the specified vector or line. (For each line, write the direction numbers as integers.)

<table>
<thead>
<tr>
<th>Point</th>
<th>Parallel to</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. ((0, 0, 0))</td>
<td>(v = (1, 2, 3))</td>
</tr>
<tr>
<td>6. ((3, -5, 1))</td>
<td>(v = (3, -7, -10))</td>
</tr>
<tr>
<td>7. ((-4, 1, 0))</td>
<td>(v = \frac{3}{2}i + \frac{1}{2}j - k)</td>
</tr>
<tr>
<td>8. ((-2, 0, 3))</td>
<td>(v = 2i + 4j - 2k)</td>
</tr>
<tr>
<td>9. ((2, -3, 5))</td>
<td>(x = 5 + 2t, y = 7 - 3t, z = -2 + t)</td>
</tr>
<tr>
<td>10. ((1, 0, 1))</td>
<td>(x = 3 + 3t, y = 5 - 2t, z = -7 + t)</td>
</tr>
</tbody>
</table>

In Exercises 11–18, find (a) a set of parametric equations and (b) if possible, a set of symmetric equations of the line that passes through the given points. (For each line, write the direction numbers as integers.)

| \((2, 0, 2), (1, 4, -3)\) | \((2, 3, 0), (10, 8, 12)\) |
| \((3, -8, 15), (1, -2, 16)\) | \((2, 3, -1), (1, -5, 3)\) |
| \((3, 1, 2), (-1, 1, 5)\) | \((2, -1, 5), (2, 1, -3)\) |
| \((-\frac{1}{2}, 2, \frac{3}{2}), (1, -\frac{5}{2}, 0)\) | \((-\frac{3}{2}, 3, 2), (3, -5, -4)\) |

In Exercises 19 and 20, sketch a graph of the line.

19. \(x = 2t, y = 2 + t, z = 1 + \frac{t}{2}\)
20. \(x = 5 - 2t, y = 1 + t, z = 5 - \frac{t}{2}\)

In Exercises 21–26, find the general form of the equation of the plane passing through the point and perpendicular to the specified vector or line.

<table>
<thead>
<tr>
<th>Point</th>
<th>Perpendicular to</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2, 1, 2))</td>
<td>(n = i)</td>
</tr>
<tr>
<td>((1, 0, -3))</td>
<td>(n = k)</td>
</tr>
<tr>
<td>((5, 6, 3))</td>
<td>(n = -2i + j - 2k)</td>
</tr>
<tr>
<td>((0, 0, 0))</td>
<td>(n = -3j + 5k)</td>
</tr>
<tr>
<td>((2, 0, 0))</td>
<td>(x = 3 - t, y = 2 - 2t, z = 4 + t)</td>
</tr>
<tr>
<td>((0, 0, 6))</td>
<td>(x = 1 - t, y = 2 + t, z = 4 - 2t)</td>
</tr>
</tbody>
</table>

In Exercises 27–30, find the general form of the equation of the plane passing through the three points.

27. \((0, 0, 0), (1, 2, 3), (-2, 3, 3)\)
28. \((4, -1, 3), (2, 5, 1), (-1, 2, 1)\)
29. \((2, 3, -2), (3, 4, 2), (1, -1, 0)\)
30. \((-5, -1, 4), (1, -1, 2), (2, 1, -3)\)

In Exercises 31–36, find the general form of the equation of the plane with the given characteristics.

31. Passes through \((2, 5, 3)\) and is parallel to the \(xz\)-plane
32. Passes through \((1, 2, 3)\) and is parallel to the \(yz\)-plane
33. Passes through \((0, 2, 4)\) and \((-1, -2, 0)\) and is perpendicular to the \(yz\)-plane
34. Passes through \((1, -2, 4)\) and \((4, 0, -1)\) and is perpendicular to the \(xz\)-plane
35. Passes through \((2, 2, 1)\) and \((-1, 1, -1)\) and is perpendicular to \(2x - 3y + z = 3\)
36. Passes through \((1, 2, 0)\) and \((-1, -1, 2)\) and is perpendicular to \(2x - 3y + z = 6\)

In Exercises 37–40, determine whether the planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.

37. \(5x - 3y + z = 4\)
38. \(3x + y - 4z = 3\)
\(x + 4y + 7z = 1\)
\(-9x - 3y + 12z = 4\)
39. \(2x - z = 1\)
40. \(x - 5y - z = 1\)
\(4x + y + 8z = 10\)
\(5x - 25y - 5z = -3\)

In Exercises 41–46, find a set of parametric equations of the line. (There are many correct answers.)

41. Passes through \((2, 3, 4)\) and is parallel to the \(xz\)-plane and the \(yz\)-plane
42. Passes through \((-4, 5, 2)\) and is parallel to the \(xy\)-plane and the \(yz\)-plane
43. Passes through \((2, 3, 4)\) and is perpendicular to \(3x + 2y - z = 6\)
In Exercises 57–60, find the distance between the point and the plane.

47. $3x - 4y + 5z = 6$
48. $x - 3y + z = -2$
49. $x + y - z = 0$
50. $2x + 4y - 2z = 1$

In Exercises 47–50, (a) find the angle between the two planes and (b) find parametric equations of their line of intersection.

51. $x + 2y + 3z = 6$
52. $2x - y + 4z = 4$
53. $x + 2y = 4$
54. $y + z = 5$
55. $3x + 2y - z = 6$
56. $x - 3z = 6$

In Exercises 57–60, find the distance between the point and the plane.

57. $(0, 0, 0)$
58. $(3, 2, 1)$
59. $(4, -2, -2)$
60. $(-1, 2, 5)$

57. $8x - 4y + z = 8$
58. $x - y + 2z = 4$

DATA ANALYSIS: BEVERAGE CONSUMPTION The table shows the per capita consumption (in gallons) of different types of beverages sold by a company from 2006 through 2010. Consumption of energy drinks, soft drinks, and bottled water are represented by the variables $x$, $y$, and $z$, respectively.

<table>
<thead>
<tr>
<th>Year</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2006</td>
<td>2.3</td>
<td>3.4</td>
<td>3.9</td>
</tr>
<tr>
<td>2007</td>
<td>2.2</td>
<td>3.2</td>
<td>3.8</td>
</tr>
<tr>
<td>2008</td>
<td>2.0</td>
<td>3.1</td>
<td>3.5</td>
</tr>
<tr>
<td>2009</td>
<td>1.9</td>
<td>3.0</td>
<td>3.4</td>
</tr>
<tr>
<td>2010</td>
<td>1.8</td>
<td>2.9</td>
<td>3.3</td>
</tr>
</tbody>
</table>

A model for the data is given by $1.54x - 0.32y - z = -1.45$.

(a) Complete a fifth column in the table using the model to approximate $z$ for the given values of $x$ and $y$.

(b) Compare the approximations from part (a) with the actual values of $z$.

(c) According to this model, any increases or decreases in consumption of two types of beverages will have what effect on the consumption of the third type of beverage?

62. MECHANICAL DESIGN A chute at the top of a grain elevator of a combine funnels the grain into a bin, as shown in the figure. Find the angle between two adjacent sides.

EXPLORATION

TRUE OR FALSE? In Exercises 63 and 64, determine whether the statement is true or false. Justify your answer.

63. Every two lines in space are either intersecting or parallel.
64. Two nonparallel planes in space will always intersect.

65. The direction numbers of two distinct lines in space are $10, -18, 20$, and $-15, 27, -30$. What is the relationship between the lines? Explain.

66. Consider the following four planes.

- $2x + 3y - z = 2$
- $4x + 6y - 2z = 5$
- $-2x - 3y + z = -2$
- $-6x - 9y + 3z = 11$

What are the normal vectors for each plane? What can you say about the relative positions of these planes in space?

67. (a) Describe and find an equation for the surface generated by all points $(x, y, z)$ that are two units from the point $(4, -1, 1)$.

(b) Describe and find an equation for the surface generated by all points $(x, y, z)$ that are two units from the plane $4x - 3y + z = 10$.

68. CAPSTONE Give the parametric equations and the symmetric equations of a line in space. Describe what is required to find these equations.
## Chapter Summary

**What Did You Learn?**

<table>
<thead>
<tr>
<th>Plot points in the three-dimensional coordinate system (p. 810).</th>
<th>1–4</th>
</tr>
</thead>
</table>
| Find distances between points in space and find midpoints of line segments joining points in space (p. 811). | The distance between the points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) given by the Distance Formula in Space is 
\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \]
The midpoint of the line segment joining the points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) given by the Midpoint Formula in Space is 
\[ \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right). \] | 5–14 |
| Write equations of spheres in standard form and find traces of surfaces in space (p. 812). | **Standard Equation of a Sphere**
The standard equation of a sphere with center \((h, k, j)\) and radius \(r\) is given by \((x - h)^2 + (y - k)^2 + (z - j)^2 = r^2.\) | 15–26 |
| Find the component forms of the unit vectors in the same direction of, the magnitudes of, the dot products of, and the angles between vectors in space (p. 817). | **Vectors in Space**
1. Two vectors are equal if and only if their corresponding components are equal.
2. Magnitude of \(\mathbf{u} = (u_1, u_2, u_3)\): 
\[ ||\mathbf{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}. \]
3. A unit vector \(\mathbf{u}\) in the direction of \(\mathbf{v}\) is 
\[ \mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}, \quad \mathbf{v} \neq 0. \]
4. The sum of \(\mathbf{u} = (u_1, u_2, u_3)\) and \(\mathbf{v} = (v_1, v_2, v_3)\) is 
\[ \mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3). \]
5. The scalar multiple of the real number \(c\) and 
\(\mathbf{u} = (u_1, u_2, u_3)\) is 
\[ c\mathbf{u} = (cu_1, cu_2, cu_3). \]
6. The dot product of \(\mathbf{u} = (u_1, u_2, u_3)\) and \(\mathbf{v} = (v_1, v_2, v_3)\) is 
\[ \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3. \]
**Angle Between Two Vectors**
If \(\theta\) is the angle between two nonzero vectors \(\mathbf{u}\) and \(\mathbf{v}\), then 
\[ \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||}. \] | 27–36 |
| Determine whether vectors in space are parallel or orthogonal (p. 819). | Two nonzero vectors \(\mathbf{u}\) and \(\mathbf{v}\) are parallel if there is some scalar \(c\) such that 
\[ \mathbf{u} = c\mathbf{v}. \] | 37–44 |
| Use vectors in space to solve real-life problems (p. 821). | Vectors can be used to solve equilibrium problems in space. (See Example 7.) | 45, 46 |
### What Did You Learn?

<table>
<thead>
<tr>
<th>Section 11.3</th>
<th>Section 11.4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>What Did You Learn?</strong></td>
<td><strong>What Did You Learn?</strong></td>
</tr>
<tr>
<td>Find cross products of vectors in space (p. 824).</td>
<td>Definition of Cross Product of Two Vectors in Space&lt;br&gt;Let ( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} ) and ( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} ) be vectors in space. The cross product of ( \mathbf{u} ) and ( \mathbf{v} ), ( \mathbf{u} \times \mathbf{v} ), is the vector ((u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k} ).</td>
</tr>
<tr>
<td>Use geometric properties of cross products of vectors in space (p. 826).</td>
<td>Geometric Properties of the Cross Product&lt;br&gt;Let ( \mathbf{u} ) and ( \mathbf{v} ) be nonzero vectors in space, and let ( \theta ) be the angle between ( \mathbf{u} ) and ( \mathbf{v} ).&lt;br&gt;1. ( \mathbf{u} \times \mathbf{v} ) is orthogonal to both ( \mathbf{u} ) and ( \mathbf{v} ).&lt;br&gt;2. ( | \mathbf{u} \times \mathbf{v} | = | \mathbf{u} | | \mathbf{v} | \sin \theta ).&lt;br&gt;3. ( \mathbf{u} \times \mathbf{v} = 0 ) if and only if ( \mathbf{u} ) and ( \mathbf{v} ) are scalar multiples of each other.&lt;br&gt;4. ( | \mathbf{u} \times \mathbf{v} | = ) area of parallelogram having ( \mathbf{u} ) and ( \mathbf{v} ) as adjacent sides.</td>
</tr>
</tbody>
</table>
| Use triple scalar products to find volumes of parallelepipeds (p. 828). | The Triple Scalar Product<br>For \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \), \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \), and<br>\( \mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k} \), the triple scalar product is given by<br>\[ \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \]
| Geometric Property of the Triple Scalar Product<br>The volume \( V \) of a parallelepiped with vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) as adjacent edges is given by \( V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \). |
| Find parametric and symmetric equations of lines in space (p. 831). | Parametric Equations of a Line in Space<br>A line \( L \) parallel to the vector \( \mathbf{v} = \langle a, b, c \rangle \) and passing through the point \( P(x_1, y_1, z_1) \) is represented by the parametric equations \( x = x_1 + at, y = y_1 + bt, \) and \( z = z_1 + ct \). |
| Find equations of planes in space (p. 833). | Standard Equation of a Plane in Space<br>The plane containing the point \( (x_1, y_1, z_1) \) and having normal vector \( \mathbf{n} = \langle a, b, c \rangle \) can be represented by the standard form of the equation of a plane<br>\( a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \). |
| Sketch planes in space (p. 836). | See Figure 11.32, which shows how to sketch the plane<br>\( 3x + 2y + 4z = 12 \). |
| Find distances between points and planes in space (p. 837). | Distance Between a Point and a Plane<br>The distance between a plane and a point \( Q \) (not in the plane) is<br>\[ D = |\text{proj}_n \mathbf{PQ}| = \frac{|\mathbf{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}, \]
where \( P \) is a point in the plane and \( \mathbf{n} \) is normal to the plane. |

### Review Exercises

<table>
<thead>
<tr>
<th>Chapter Summary</th>
<th>Exercises</th>
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</thead>
<tbody>
<tr>
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<tr>
<td>841</td>
<td>51–56</td>
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<tr>
<td>841</td>
<td>57, 58</td>
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<tr>
<td>841</td>
<td>59–62</td>
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<tr>
<td>841</td>
<td>63–66</td>
</tr>
<tr>
<td>841</td>
<td>67–70</td>
</tr>
<tr>
<td>841</td>
<td>71–74</td>
</tr>
</tbody>
</table>
In Exercises 1–2, plot each point in the same three-dimensional coordinate system.
1. (a) (5, -1, 2)       2. (a) (2, 4, -3)
   (b) (-3, 3, 0)       (b) (0, 0, 5)

3. Find the coordinates of the point in the xy-plane four units to the right of the xz-plane and five units behind the yz-plane.
4. Find the coordinates of the point located on the y-axis and seven units to the left of the xz-plane.

In Exercises 5–8, find the distance between the points.
5. (4, 0, 7), (5, 2, 1)  6. (2, 3, -4), (-1, -3, 0)
7. (-7, -5, 6), (1, 1, 6)  8. (0, 0, 0), (4, 4, 4)

In Exercises 9 and 10, find the lengths of the sides of the right triangle. Show that these lengths satisfy the Pythagorean Theorem.
9. 
10. 

In Exercises 11–14, find the midpoint of the line segment joining the points.
11. (8, -2, 3), (5, 6, 7) 12. (7, 1, -4), (1, -1, 2)
13. (10, 6, -12), (-8, -2, -6) 14. (-5, -3, 1), (-7, -9, -5)

In Exercises 15–20, find the standard form of the equation of the sphere with the given characteristics.
15. Center: (2, 3, 5); radius: 1
16. Center: (3, -2, 4); radius: 4
17. Center: (1, 5, 2); diameter: 12
18. Center: (0, 4, -1); diameter: 15
19. Endpoints of a diameter: (-2, -2, -2), (2, 2, 2)
20. Endpoints of a diameter: (4, -1, -3), (-2, 5, 3)

In Exercises 21–24, find the center and radius of the sphere.
21. \(x^2 + y^2 + z^2 - 8z = 0\)
22. \(x^2 + y^2 + z^2 - 4x - 6y + 4 = 0\)
23. \(x^2 + y^2 + z^2 - 10x + 6y - 4z + 34 = 0\)
24. \(2x^2 + 2y^2 + 2z^2 + 2x + 2y + 2z + 1 = 0\)

In Exercises 25 and 26, sketch the graph of the equation and sketch the specified trace.
25. \(x^2 + (y - 3)^2 + z^2 = 16\)
   (a) xz-trace
   (b) yz-trace
26. \((x + 2)^2 + (y - 1)^2 + z^2 = 9\)
   (a) xy-trace
   (b) yz-trace

In Exercises 27–30, (a) write the component form of the vector \(v\), (b) find the magnitude of \(v\), and (c) find a unit vector in the direction of \(v\).
27. Initial point: (3, -2, 1)
   Terminal point: (4, 4, 0)
28. Initial point: (2, -1, 2)
   Terminal point: (-3, 2, 3)
29. Initial point: (7, -4, 3)
   Terminal point: (-3, 2, 10)
30. Initial point: (0, 3, -1)
   Terminal point: (5, -8, 6)

In Exercises 31–34, find the dot product of \(u\) and \(v\).
31. \(u = (-1, 4, 3)\)  \(v = (0, -6, 5)\)  \(v = (2, 5, 2)\)
32. \(u = (8, -4, 2)\)
33. \(u = 2i - j + k\)  \(v = i - k\)
34. \(u = 2i + j - 2k\)  \(v = i - 3j + 2k\)

In Exercises 35 and 36, find the angle \(\theta\) between the vectors.
35. \(u = (2, -1, 0)\)  \(v = (1, 2, 1)\)
36. \(u = (3, 1, -1)\)  \(v = (4, 5, 2)\)

In Exercises 37–40, determine whether \(u\) and \(v\) are orthogonal, parallel, or neither.
37. \(u = (39, -12, 21)\)  \(v = (-26, 8, -14)\)
38. \(u = (8, 5, -8)\)  \(v = (-2, 4, 0)\)
39. \(u = 6i + 5j + 9k\)  \(v = 5i + 3j - 5k\)
40. \(u = 3j + 2k\)  \(v = 12i - 18k\)
In Exercises 41–44, use vectors to determine whether the points are collinear.

41. (6, 3, –1), (5, 8, 3), (7, –2, –5)
42. (5, 2, 0), (2, 6, 1), (2, 4, 7)
43. (5, –4, 7), (8, –5, 5), (11, 6, 3)
44. (3, 4, –1), (–1, 6, 9), (5, 3, –6)

45. **TENSION** A load of 300 pounds is supported by three cables, as shown in the figure. Find the tension in each of the supporting cables.

![Diagram of cables](image)

46. **TENSION** Determine the tension in each of the supporting cables in Exercise 45 if the load is 200 pounds.

**11.3** In Exercises 47–50, find \( u \times v \).

47. \( u = \langle -2, 8, 2 \rangle \) \( v = \langle 1, 1, -1 \rangle \)
48. \( u = \langle 10, 15, 5 \rangle \) \( v = \langle 5, -3, 0 \rangle \)
49. \( u = 2i + 3j + 2k \) \( v = 3i + j + 2k \)
50. \( u = -i + 2j - 2k \) \( v = i \)

In Exercises 51–54, find a unit vector orthogonal to \( u \) and \( v \).

51. \( u = 2i - j + k \) \( v = -i + j - 2k \)
52. \( u = j + 4k \) \( v = -i + 3j \)
53. \( u = -3i + 2j - 5k \) \( v = 10i - 15j + 2k \)
54. \( u = 4k \) \( v = i + 12k \)

In Exercises 55 and 56, verify that the points are the vertices of a parallelogram and find its area.

55. (2, –1, 1), (5, 1, 4), (0, 1, 1), (3, 3, 4)
56. (0, 4, 0), (1, 4, 1), (0, 6, 0), (1, 6, 1)

In Exercises 57 and 58, find the volume of the parallelepiped with the given vertices.

57. \( A(0, 0, 0), B(3, 0, 0), C(0, 5, 1), D(3, 5, 1), E(2, 0, 5), F(5, 0, 5), G(2, 5, 6), H(5, 5, 6) \)
58. \( A(0, 0, 0), B(2, 0, 0), C(2, 4, 0), D(0, 4, 0), E(0, 0, 6), F(2, 0, 6), G(2, 4, 6), H(0, 4, 6) \)

**11.4** In Exercises 59–62, find a set of (a) parametric equations and (b) symmetric equations for the specified line.

59. Passes through \((-1, 3, 5)\) and \((3, 6, -1)\)
60. Passes through \((0, -10, 3)\) and \((5, 10, 0)\)
61. Passes through \((0, 0, 0)\) and is parallel to \( v = \langle -2, \frac{1}{2}, 1 \rangle \)
62. Passes through \((3, 2, 1)\) and is parallel to the line given by \( x = y = z \)

In Exercises 63–66, find the general form of the equation of the specified plane.

63. Passes through \((0, 0, 0)\), \((5, 0, 2)\), and \((2, 3, 8)\)
64. Passes through \((-1, 3, 4)\), \((4, -2, 2)\), and \((2, 8, 6)\)
65. Passes through \((5, 3, 2)\) and is parallel to the \(xy\)-plane
66. Passes through \((0, 0, 6)\) and is perpendicular to the line given by \( x = 1 - t, y = 2 + t, \) and \( z = 4 - 2t \)

In Exercises 67–70, plot the intercepts and sketch a graph of the plane.

67. \( 3x - 2y + 3z = 6 \)
68. \( 5x - y - 5z = 5 \)
69. \( 2x - 3z = 6 \)
70. \( 4y - 3z = 12 \)

In Exercises 71–74, find the distance between the point and the plane.

71. \((1, 2, 3)\)
72. \((2, 3, 10)\)
73. \((0, 0, 0)\)
74. \((0, 0, 0)\)

\[ 2x - y + z = 4 \quad x - 10y + 3z = 3 \]
\[ 2x + 3y + z = 12 \quad x - 10y + 3z = 2 \]

**EXPLORATION**

**TRUE OR FALSE?** In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.

75. The cross product is commutative.
76. The triple scalar product of three vectors in space is a scalar.

In Exercises 77 and 78, let \( u = \langle u_1, u_2, u_3 \rangle, v = \langle v_1, v_2, v_3 \rangle, \) and \( w = \langle w_1, w_2, w_3 \rangle \).

77. Show that \( \langle u + v \rangle = u \cdot v + u \cdot w \).
78. Show that \( \langle u \times v \rangle = (u \times v) + (u \times w) \).
Chapter 11 Analytic Geometry in Three Dimensions

11. Chapter Test

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Plot each point in the same three-dimensional coordinate system.
   (a) \((3, -7, 2)\)
   (b) \((2, 2, -1)\)

2. Consider the triangle with vertices \(A, B,\) and \(C\). Is it a right triangle? Explain.
3. Find the coordinates of the midpoint of the line segment joining points \(A\) and \(B\).
4. Find the standard form of the equation of the sphere for which \(A\) and \(B\) are the endpoints of a diameter. Sketch the sphere and its \(xz\)-trace.

In Exercises 5–9, let \(u\) and \(v\) be the vectors from \(A(8, -2, 5)\) to \(B(6, 4, -1)\) and from \(A\) to \(C(-4, 3, 0)\), respectively.

5. Write \(u\) and \(v\) in component form.
6. Find \((a)\) \(u \cdot v\) and \((b)\) \(u \times v\).
7. Find \((a)\) a unit vector in the direction of \(u\) and \((b)\) a unit vector in the direction of \(v\).
8. Find the angle between \(u\) and \(v\).
9. Find a set of \((a)\) parametric equations and \((b)\) symmetric equations for the line through points \(A\) and \(B\).

In Exercises 10–12, determine whether \(u\) and \(v\) are orthogonal, parallel, or neither.

10. \(u = i - 2j - k\)
11. \(u = -3i + 2j - k\)
12. \(u = \langle 4, -1, 6 \rangle\)
    \(v = j + 6k\)
    \(v = i - j - k\)
    \(v = \langle -2, \frac{1}{2}, -3 \rangle\)

13. Verify that the points \(A(2, -3, 1), B(6, 5, -1), C(3, -6, 4),\) and \(D(7, 2, 2)\) are the vertices of a parallelogram, and find its area.
14. Find the volume of the parallelepiped at the left with the given vertices.
    \(A(0, 0, 5), B(0, 10, 5), C(4, 10, 5), D(4, 0, 5), E(0, 1, 0), F(0, 11, 0), G(4, 11, 0), H(4, 1, 0)\)

In Exercises 15 and 16, plot the intercepts and sketch a graph of the plane.

15. \(3x + 6y + 2z = 18\)
16. \(5x - y - 2z = 10\)

17. Find the general form of the equation of the plane passing through the points \((-3, -4, 2), (-3, 4, 1),\) and \((1, 1, -2)\).
18. Find the distance between the point \((2, -1, 6)\) and the plane \(3x - 2y + z = 6\).
19. A tractor fuel tank has the shape and dimensions shown in the figure. In fabricating the tank, it is necessary to know the angle between two adjacent sides. Find this angle.
**Notation for Dot and Cross Products**

The notation for the dot product and the cross product of vectors was first introduced by the American physicist Josiah Willard Gibbs (1839–1903). In the early 1880s, Gibbs built a system to represent physical quantities called vector analysis. The system was a departure from William Hamilton’s theory of quaternions.

**Algebraic Properties of the Cross Product** *(p. 825)*

Let \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) be vectors in space and let \( c \) be a scalar.

1. \( \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \)
2. \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \)
3. \( c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) \)
4. \( \mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0} \)
5. \( \mathbf{u} \times \mathbf{u} = \mathbf{0} \)
6. \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \)

**Proof**

Let \( \mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k} \), \( \mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \), and let \( c \) be a scalar.

1. \( \mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \)
2. \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \)
3. \( c(\mathbf{u} \times \mathbf{v}) = c(u_2v_3 - u_3v_2)\mathbf{i} - c(u_1v_3 - u_3v_1)\mathbf{j} + c(u_1v_2 - u_2v_1)\mathbf{k} \)
4. \( \mathbf{u} \times \mathbf{0} = (u_2 \cdot 0 - u_3 \cdot 0)\mathbf{i} - (u_1 \cdot 0 - u_3 \cdot 0)\mathbf{j} + (u_1 \cdot 0 - u_2 \cdot 0)\mathbf{k} \)
5. \( \mathbf{u} \times \mathbf{u} = (u_2u_3 - u_3u_2)\mathbf{i} - (u_1u_3 - u_3u_1)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} \)
6. \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \)

\( (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \)

\( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - w_1v_3) + u_3(v_1w_2 - w_1v_2) \)

\( = u_1v_2w_3 - u_1w_2v_3 - u_2v_1w_3 + u_2w_1v_3 + u_3v_1w_2 - u_3w_1v_2 \)

\( = u_2v_1w_3 - u_2w_1v_3 - u_3v_1w_2 + u_3w_1v_2 - u_1v_2w_3 + u_1w_2v_3 \)

\( = w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1) \)

\( = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \)
**Geometric Properties of the Cross Product (p. 826)**

Let \( \mathbf{u} \) and \( \mathbf{v} \) be nonzero vectors in space, and let \( \theta \) be the angle between \( \mathbf{u} \) and \( \mathbf{v} \).

1. \( \mathbf{u} \times \mathbf{v} \) is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).
2. \( \| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta \)
3. \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \) if and only if \( \mathbf{u} \) and \( \mathbf{v} \) are scalar multiples of each other.
4. \( \| \mathbf{u} \times \mathbf{v} \| = \text{area of parallelogram having } \mathbf{u} \text{ and } \mathbf{v} \text{ as adjacent sides.}

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**Proof**

Let \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \), \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \), and \( \mathbf{0} = 0 \mathbf{i} + 0 \mathbf{j} + 0 \mathbf{k} \).

1. \( \mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k} \)
   
   \[ \begin{align*}
   (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= (u_2 v_3 - u_3 v_2) u_1 - (u_1 v_3 - u_3 v_1) u_2 + (u_1 v_2 - u_2 v_1) u_3 \\
   &= u_1 u_2 v_3 - u_1 u_3 v_2 - u_2 u_3 v_1 + u_2 u_3 v_1 + u_1 u_3 v_2 - u_3 u_2 v_1 = 0 \\
   (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} &= (u_2 v_3 - u_3 v_2) v_1 - (u_1 v_3 - u_3 v_1) v_2 + (u_1 v_2 - u_2 v_1) v_3 \\
   &= u_2 v_1 v_3 - u_1 v_1 v_2 - u_2 v_2 v_3 + u_3 v_1 v_2 + u_3 v_2 v_1 - u_3 v_1 v_2 = 0
   \end{align*} \]

Because two vectors are orthogonal if their dot product is zero, it follows that \( \mathbf{u} \times \mathbf{v} \) is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).

2. Note that \( \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \). So,

   \[
   \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta = \| \mathbf{u} \| \| \mathbf{v} \| \sqrt{1 - \cos^2 \theta} \\
   = \| \mathbf{u} \| \| \mathbf{v} \| \sqrt{1 - \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} \right)^2} \\
   = \sqrt{\| \mathbf{u} \|^2 \| \mathbf{v} \|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\
   = \sqrt{( u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2} \\
   = \sqrt{ (u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2} = \| \mathbf{u} \times \mathbf{v} \|
   \]

3. If \( \mathbf{u} \) and \( \mathbf{v} \) are scalar multiples of each other, then \( \mathbf{u} = c \mathbf{v} \) for some scalar \( c \).

   \[ \mathbf{u} \times \mathbf{v} = (c \mathbf{v}) \times \mathbf{v} = c (\mathbf{v} \times \mathbf{v}) = c \mathbf{0} = \mathbf{0} \]

   If \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \), then \( \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta = 0 \). (Assume \( \mathbf{u} \neq \mathbf{0} \) and \( \mathbf{v} \neq \mathbf{0} \).) So, \( \sin \theta = 0 \), and \( \theta = 0 \) or \( \theta = \pi \). In either case, because \( \theta \) is the angle between the vectors, \( \mathbf{u} \) and \( \mathbf{v} \) are parallel. So, \( \mathbf{u} = c \mathbf{v} \) for some scalar \( c \).

4. The figure at the left is a parallelogram having \( \mathbf{v} \) and \( \mathbf{u} \) as adjacent sides. Because the height of the parallelogram is \( \| \mathbf{v} \| \sin \theta \), the area is

   \[ \text{Area} = (\text{base})(\text{height}) = \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta = \| \mathbf{u} \times \mathbf{v} \|. \]
PROBLEM SOLVING

This collection of thought-provoking and challenging exercises further explores and expands upon concepts learned in this chapter.

1. Let \( \mathbf{u} = i + j \), \( \mathbf{v} = j + k \), and \( \mathbf{w} = a \mathbf{u} + b \mathbf{v} \).
   (a) Sketch \( \mathbf{u} \) and \( \mathbf{v} \).
   (b) If \( \mathbf{w} = 0 \), show that \( a \) and \( b \) must both be zero.
   (c) Find \( a \) and \( b \) such that \( \mathbf{w} = i + 2j + k \).
   (d) Show that no choice of \( a \) and \( b \) yields \( \mathbf{w} = i + 2j + 3k \).

2. The initial and terminal points of \( \mathbf{v} \) are \( (x_1, y_1, z_1) \) and \( (x, y, z) \), respectively. Describe the set of all points \( (x, y, z) \) such that \( \| \mathbf{v} \| = 4 \).

3. You are given the component forms of the vectors \( \mathbf{u} \) and \( \mathbf{v} \). Write a program for a graphing utility in which the output is (a) the component form of \( \mathbf{u} + \mathbf{v} \), (b) \( \| \mathbf{u} + \mathbf{v} \| \), (c) \( \| \mathbf{u} \| \), and (d) \( \| \mathbf{v} \| \).

4. Run the program you wrote in Exercise 3 for the vectors \( \mathbf{u} = (-1, 3, 4) \) and \( \mathbf{v} = (5, 4.5, -6) \).

5. The vertices of a triangle are given. Determine whether the triangle is an acute triangle, an obtuse triangle, or a right triangle. Explain your reasoning.
   (a) \( (1, 2, 0), (0, 0, 0), (-2, 1, 0) \)
   (b) \( (-3, 0, 0), (0, 0, 0), (1, 2, 3) \)
   (c) \( (2, -3, 4), (0, 1, 2), (-1, 2, 0) \)
   (d) \( (2, -7, 3), (-1, 5, 8), (4, 6, -1) \)

6. A television camera weighing 120 pounds is supported by a tripod (see figure). Represent the force exerted on each leg of the tripod as a vector.

7. A precast concrete wall is temporarily kept in its vertical position by ropes (see figure). Find the total force exerted on the pin at position \( A \). The tensions in \( AB \) and \( AC \) are 420 pounds and 650 pounds, respectively.

8. Prove \( \| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \) if \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal.

9. Prove \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} \).

10. Prove that the triple scalar product of \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) is given by 
    \[ \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} . \]

11. Prove that the volume \( V \) of a parallelepiped with vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) as adjacent edges is given by 
    \[ V = \| \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \| . \]

12. In physics, the cross product can be used to measure torque, or the moment \( \mathbf{M} \) of a force \( \mathbf{F} \) about a point \( P \). If the point of application of the force is \( Q \), the moment of \( \mathbf{F} \) about \( P \) is given by \( \mathbf{M} = \overrightarrow{PQ} \times \mathbf{F} \). A force of 60 pounds acts on the pipe wrench shown in the figure.

(a) Find the magnitude of the moment about \( O \). Use a graphing utility to graph the resulting function of \( \theta \).
(b) Use the result of part (a) to determine the magnitude of the moment when \( \theta = 45^\circ \).
(c) Use the result of part (a) to determine the angle \( \theta \) when the magnitude of the moment is maximum. Is the answer what you expected? Why or why not?
15. A force of 200 pounds acts on the bracket shown in the figure.

\[ \begin{array}{c}
\text{F} \\
200 \text{ lb}
\end{array} \]

(a) Determine the vector \( \overrightarrow{AB} \) and the vector \( \mathbf{F} \) representing the force. (\( \mathbf{F} \) will be in terms of \( \theta \).

(b) Find the magnitude of the moment (torque) about \( A \) by evaluating \( \| \overrightarrow{AB} \times \mathbf{F} \|. \) Use a graphing utility to graph the resulting function of \( \theta \) for \( 0^\circ \leq \theta \leq 180^\circ \).

(c) Use the result of part (b) to determine the magnitude of the moment when \( \theta = 30^\circ \).

(d) Use the result of part (b) to determine the angle \( \theta \) when the magnitude of the moment is maximum.

(e) Use the graph in part (b) to approximate the zero of the function. Interpret the meaning of the zero in the context of the problem.

14. Using vectors, prove the Law of Sines: If \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) are three sides of the triangle shown in the figure, then

\[
\frac{\sin A}{\| \mathbf{a} \|} = \frac{\sin B}{\| \mathbf{b} \|} = \frac{\sin C}{\| \mathbf{c} \|}.
\]

15. Two insects are crawling along different lines in three-space. At time \( t \) (in minutes), the first insect is at the point \( (x, y, z) \) on the line given by

\[
x = 6 + t, \quad y = 8 - t, \quad z = 3 + t.
\]

Also, at time \( t \), the second insect is at the point \( (x, y, z) \) on the line given by

\[
x = 1 + t, \quad y = 2 + t, \quad z = 2t.
\]

Assume distances are given in inches.

(a) Find the distance between the two insects at time \( t = 0 \).

(b) Use a graphing utility to graph the distance between the insects from \( t = 0 \) to \( t = 10 \).

(c) Using the graph from part (b), what can you conclude about the distance between the insects?

(d) Using the graph from part (b), determine how close the insects get to each other.

16. The distance between a point \( Q \) and a line in space is given by

\[
D = \frac{\| \overrightarrow{PQ} \times \mathbf{u} \|}{\| \mathbf{u} \|}
\]

where \( \mathbf{u} \) is a direction vector for the line and \( \mathbf{P} \) is a point on the line. Find the distance between the point and the line given by each set of parametric equations.

(a) \((1, 5, -2)\)

\[
x = -2 + 4t, \quad y = 3, \quad z = 1 - t
\]

(b) \((1, -2, 4)\)

\[
x = 2t, \quad y = -3 + t, \quad z = 2 + 2t
\]

17. Use the formula given in Exercise 16.

(a) Find the shortest distance between the point \( Q(2, 0, 0) \) and the line determined by the points \( P_1(0, 0, 1) \) and \( P_2(0, 1, 2) \).

(b) Find the shortest distance between the point \( Q(2, 0, 0) \) and the line segment joining the points \( P_1(0, 0, 1) \) and \( P_2(0, 1, 2) \).

18. Consider the line given by the parametric equations

\[
x = -t + 3, \quad y = \frac{1}{2}t + 1, \quad z = 2t - 1
\]

and the point \((4, 3, s)\) for any real number \( s \).

(a) Write the distance between the point and the line as a function of \( s \). (Hint: Use the formula given in Exercise 16.)

(b) Use a graphing utility to graph the function from part (a). Use the graph to find the value of \( s \) such that the distance between the point and the line is a minimum.

(c) Use the zoom feature of the graphing utility to zoom out several times on the graph in part (b). Does it appear that the graph has slant asymptotes? Explain. If it appears to have slant asymptotes, find them.